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ORIGINAL ARTICLE

Bodily tides near spin–orbit resonances

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Abstract Spin–orbit coupling can be described in two approaches. The first method, known as the *"MacDonald torque"*, is often combined with a convenient assumption that the quality factor Q is frequency-independent. This makes the method inconsistent, because derivation of the expression for the MacDonald torque tacitly fixes the rheology of the mantle by making Q scale as the inverse tidal frequency. Spin–orbit coupling can be treated also in an approach called *"the Darwin torque"*. While this theory is general enough to accommodate an arbitrary frequency-dependence of Q, this advantage has not yet been fully exploited in the literature, where Q is often assumed constant or is set to scale as inverse tidal frequency, the latter assertion making the Darwin torque equivalent to a corrected version of the MacDonald torque. However neither a constant nor an inverse-frequency Q reflect the properties of realistic mantles and crusts, because the actual frequency-dependence is more complex. Hence it is necessary to enrich the theory of spin–orbit interaction with the right frequency-dependence. We accomplish this programme for the Darwin-torque-based model near resonances. We derive the frequency-dependence of the tidal torque from the first principles of solid-state mechanics, i.e., from the expression for the mantle's compliance in the time domain. We also explain that the tidal torque includes not only the customary, secular part, but also an oscillating part. We demonstrate that the $lmpq$ term of the Darwin–Kaula expansion for the tidal torque smoothly passes zero, when the secondary traverses the $lmpq$ resonance (e.g., the principal tidal torque smoothly goes through nil as the secondary crosses the synchronous orbit). Thus, we prepare a foundation for modeling entrapment of a despinning primary into a resonance with its secondary. The roles of the primary and secondary may be played, e.g., by Mercury and the Sun, correspondingly, or by an icy moon and a Jovian planet. We also offer a possible explanation for the "improper" frequency-dependence of the tidal dissipation rate in the Moon, discovered by LLR.

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1 Introduction

The paper addresses tidal torques acting on a body in the vicinity of a spin–orbit resonance. Although the topic has been popular since mid-sixties and has already been addressed in books, the common models are not entirely adequate to the actual physics. Just as in the nonresonant case discussed in Efroimsky and Williams (2009), a generic problem with the popular models of libration or of capture into a resonance is that they employ wrong rheologies (the work by Rambaux et al. 2010 being the only exception we know of). Above that, the model based on the MacDonald torque suffers a defect stemming from a genuine inconsistency inherent in the theory by MacDonald (1964) and Gerstenkorn (1955).

As explained in Efroimsky and Williams (2009) and Williams and Efroimsky (2012), the MacDonald theory, both in its original and corrected versions, tacitly fixes an unphysical shape of the functional dependence $Q(\chi)$, where Q is the dissipation quality factor and χ is the tidal frequency. So we base our treatment on the developments by Darwin (1879, 1880) and Kaula (1964), combining those with a realistic frequency-dependence of the damping rate.

Since our main purpose is to lay the groundwork for the subsequent study of the process of falling into a resonance, the two principal results obtained in this paper are the following:

(a) Starting with a realistic rheological model (an expression for the compliance in the time domain), we derive the complex Love numbers $\bar{k}_l$ as functions of the frequency χ, and write down their negative imaginary parts as functions of the frequency: $-\mathcal{I}m\left[\bar{k}_l(\chi)\right] = |k_l(\chi)| \sin \epsilon_l(\chi)$. It is these expressions that appear as factors in the terms of the Darwin-Kaula expansion of tides. These factors' frequency-dependencies demonstrate a nontrivial shape, especially near resonances. This shape plays a crucial role in modeling of despinning in general, specifically in modeling the process of falling into a spin–orbit resonance.

(b) We demonstrate that, beside the customary secular part, the Darwin torque contains a usually omitted oscillating part.

A much extended version of this paper is available online (arXiv:1105.6086).

2 Linear bodily tides

Linearity of tide means that: (a) under a static load, deformation scales linearly, and (b) under undulatory loading, the same linear law applies, separately, to each frequency mode. Thus the deformation magnitude at each frequency should depend linearly upon the tidal stress at this frequency, and should bear no dependence upon loading at other tidal modes. So the dissipation rate at that frequency will depend on the stress at that frequency only.

2.1 Linearity of the tidal deformation

At a point $\mathbf{R} = (R, \lambda, \phi)$, the potential due to a tide-raising secondary of mass M^*_{sec}, located at $\mathbf{r}^* = (r^*, \lambda^*, \phi^*)$ with $r^* \geq R$, is expandable over the Legendre polynomials $P_l(\cos \gamma)$:

$$W(\mathbf{R}, \mathbf{r}^*) = \sum_{l=2}^{\infty} W_l(\mathbf{R}, \mathbf{r}^*) = -\frac{GM_{sec}^*}{r^*} \sum_{l=2}^{\infty} \left(\frac{R}{r^*}\right)^l P_l(\cos\gamma)$$
$$= -\frac{GM_{sec}^*}{r^*} \sum_{l=2}^{\infty} \left(\frac{R}{r^*}\right)^l \sum_{m=0}^{l} \frac{(l-m)!}{(l+m)!} (2-\delta_{0m}) P_{lm}(\sin\phi) P_{lm}(\sin\phi^*)$$
$$\times \cos m(\lambda - \lambda^*), \tag{1}$$

where $G = 6.7 \times 10^{-11}\,\mathrm{m}^3\,\mathrm{kg}^{-1}\,\mathrm{s}^{-2}$ is Newton's gravity constant, and γ is the angular separation between the vectors $\mathbf{r}^*$ and $\mathbf{R}$ pointing from the primary's centre. The latitudes $\phi,\ \phi^*$ are reckoned from the primary's equator, while the longitudes $\lambda,\ \lambda^*$ are reckoned from a fixed meridian.

Under the assumption of linearity, the term $W_l(\mathbf{R},\ \mathbf{r}^*)$ in the secondary's potential causes a linear deformation of the primary's shape. The subsequent adjustment of the primary's potential being linear in the said deformation, the adjustment U_l of this potential is proportional to W_l. The theory of potential requires $U_l(\mathbf{r})$ to fall off, outside the primary, as $r^{-(l+1)}$. Thus the overall amendment to the potential of the primary amounts to:

$$U(\mathbf{r}) = \sum_{l=2}^{\infty} U_l(\mathbf{r}) = \sum_{l=2}^{\infty} k_l \left(\frac{R}{r}\right)^{l+1} W_l(\mathbf{R},\ \mathbf{r}^*), \tag{2}$$

R now being the mean equatorial radius of the primary, $\mathbf{R} = (R, \phi, \lambda)$ being a surface point, $\mathbf{r} = (r, \phi, \lambda)$ being an exterior point located above it at a radius $r \geq R$. The coefficients k_l, called Love numbers, are defined by the primary's rheology.

For a homogeneous incompressible spherical primary of density ρ, surface gravity g, and rigidity μ, the *static* Love number of degree l is given by

$$k_l = \frac{3}{2\,(l-1)}\,\frac{1}{1+A_l}, \quad \text{where} \quad A_l \equiv \frac{(2\,l^2+4l+3)\mu}{l\,\mathrm{g}\,\rho\,R} = \frac{3\,(2\,l^2+4\,l+3)\,\mu}{4\,l\,\pi\,G\,\rho^2\,R^2}. \tag{3}$$

For $R \ll r, r^*$, consideration of the $l = 2$ input in (2) turns out to be sufficient.[1]

These formulae apply to *static* deformations. However an actual tide is never static, except in the case of synchronous orbiting with a zero eccentricity and inclination.[2] A realistic perturbing potential produced by the secondary carries a spectrum of modes ω_{lmpq} (positive or negative) numbered with four integers $lmpq$ as in formula (101) below. The perturbation causes a spectrum of stresses in the primary, at frequencies $\chi_{lmpq} = |\omega_{lmpq}|$. Although in a linear medium strains are generated at the frequencies of the stresses, friction makes each Fourier component of the strain fall behind the appropriate component of the stress. Friction also reduces the magnitude of the shape response—hence the deviation of a dynamical Love number $k_l(\chi)$ from its static counterpart $k_l = k_l(0)$. Below we shall explain that formulae (2–3) can be easily adjusted to the case of undulatory tidal loads in a homogeneous planet or in tidally-despinning homogeneous satellite (treated now as the primary, with its planet playing the role of the tide-raising secondary). However generalisation of formulae (2–3) to the case of a librating moon (treated as a primary) turns out to be nontrivial, because the

[1] Special is the case of Phobos, for whose orbital evolution the k_3 and perhaps even the k_4 terms may be relevant (Bills et al. 2005). Another class of exceptions is constituted by close binary asteroids. The topic is addressed by Taylor and Margot (2010), who took into account the Love numbers up to k_6.

[2] The case of a permanently deformed moon in a 1:1 spin–orbit resonance falls under this description too. Recall that in the tidal context the distorted body is taken to be the primary. So from the viewpoint of the satellite its host planet is orbiting the satellite synchronously, thus creating a static tide.

standard derivation by Love (1909, 1911) falls apart in the presence of the non-potential inertial force containing the time-derivative of the primary's angular velocity.

The frequency-dependence of a dynamical Love number takes its origins in the "inertia" of strain and, therefore, of the shape of the body. Hence the analogy to linear circuits: the l^{th} components of W and U act as a current and voltage, while the l^{th} Love number plays, up to a factor, the role of impedance. Therefore, under a sinusoidal load of frequency χ, it is convenient to replace the actual Love number with its complex counterpart

$$\bar{k}_l(\chi) = |\bar{k}_l(\chi)| \exp\left[-i\,\epsilon_l(\chi)\right], \tag{4}$$

ϵ_l being the phase delay of the reaction relative to the load (Munk and MacDonald 1960; Zschau 1978). The "minus" sign in (4) makes U lag behind W for a positive ϵ_l. (So the situation resembles a circuit with a capacitor, where the current leads the voltage.)

For a steady deformation, the lag should vanish, and so should the entire imaginary part:

$$\mathcal{I}m\left[\bar{k}_l(0)\right] = |\,\bar{k}_l(0)|\sin\epsilon_l(0) = 0, \tag{5}$$

leaving the complex Love number real:

$$\bar{k}_l(0) = \mathcal{R}e\left[\bar{k}_l(0)\right] = |\,\bar{k}_l(0)|\cos\epsilon_l(0), \tag{6}$$

and equal to the customary static Love number:

$$\bar{k}_l(0) = k_l. \tag{7}$$

Combined with the constitutive (rheological) law, the equation of motion renders the complex $\bar{k}_l(\chi)$, as explained in Appendix B.1. Once $\bar{k}_l(\chi)$ is found, its absolute value

$$k_l(\chi) \equiv |\bar{k}_l(\chi)| \tag{8}$$

and negative argument

$$\epsilon_l(\chi) = -\arctan\frac{\mathcal{I}m\left[\bar{k}_l(\chi)\right]}{\mathcal{R}e\left[\bar{k}_l(\chi)\right]} \tag{9}$$

should be inserted into the l^{th} term of the Fourier expansion for the tidal potential. Things get simplified when we study how the tide, caused on the primary by a secondary, is acting on that same secondary. In this case, the l^{th} term in the Fourier expansion contains $|k_l(\chi)|$ and $\epsilon_l(\chi)$ in the convenient combination $k_l(\chi)\,\sin\epsilon_l(\chi)$, which is exactly $-\mathcal{I}m\left[\bar{k}_l(\chi)\right]$.

Rigorously speaking, we should say not "the l^{th} term", but "the l^{th} terms", as each l corresponds to an infinite set of positive and negative Fourier modes ω_{lmpq}, the physical forcing frequencies being $\chi = \chi_{lmpq} \equiv |\omega_{lmpq}|$. Thus, while the functional forms of both $|k_l(\chi)|$ and $\sin\epsilon_l(\chi)$ depend only on l, these functions take values that are different for different sets of numbers mpq, because χ assumes different values χ_{lmpq} on these sets. For triaxial bodies the functional forms of $|k_l(\chi)|$ and $\sin\epsilon_l(\chi)$ may depend also on $m,\ p,\ q$.

2.2 Damping of a linear tide

Besides the standard assumption $U_l(\mathbf{r}) \propto W_l(\mathbf{R},\ \mathbf{r}^*)$, the linearity condition includes the requirement that the functions $k_l(\chi)$ and $\epsilon_l(\chi)$ be well defined. This implies that they depend solely upon the frequency χ, not upon the other frequencies involved. Nor shall the Love numbers or lags be influenced by the stress or strain magnitudes at this or other frequencies.

Then, at frequency χ, the mean (over a period) damping rate $\langle \dot{E}\ (\chi)\rangle$ depends on the value of χ and on the loading at that frequency, and is not influenced by the other frequencies:

$$\langle \dot{E}(\chi)\rangle = -\chi E_{peak}(\chi)/Q(\chi) \tag{10}$$

or, equivalently:

$$\Delta E_{cycle}(\chi) = -2\pi\ E_{peak}(\chi)/Q(\chi), \tag{11}$$

$\Delta E_{cycle}(\chi)$ being the one-cycle energy loss, and $Q(\chi)$ being the so-called quality factor.

If $E_{peak}(\chi)$ in (10–11) is agreed to denote the peak *energy* stored at frequency χ, the appropriate Q factor is connected to the phase lag $\epsilon(\chi)$ through

$$Q^{-1}_{energy} = \sin|\epsilon|. \tag{12}$$

and *not* through $Q^{-1}_{energy} = \tan|\epsilon|$ as often presumed (see Appendix A for explanation).

If $E_{peak}(\chi)$ is defined as the peak *work*, the appropriate Q factor is related to the lag via

$$Q^{-1}_{work} = \tan|\epsilon| \left[1 - \left(\frac{\pi}{2} - |\epsilon|\right)\tan|\epsilon|\right]^{-1}, \tag{13}$$

as demonstrated in Appendix A below.[3] In the limit of a small ϵ, (13) becomes

$$Q^{-1}_{work} = \sin|\epsilon| + O(\epsilon^2) = |\epsilon| + O(\epsilon^2), \tag{14}$$

so definition (13) makes $1/Q$ a good approximation to $\sin\epsilon$ for small lags only.

For the lag approaching $\pi/2$, the quality factor defined through (12) attains its minimum, $Q_{energy} = 1$, while definition (13) furnishes $Q_{work} = 0$. The latter is not surprising, as in the said limit no work is carried out on the system.

Linearity requires $\bar{k}_l(\chi)$ and therefore $\epsilon_l(\chi)$ to be well-defined functions, i.e., to be independent from all the other frequencies but χ. As we see, the requirement extends to $Q(\chi)$.

The third definition, $Q^{-1}_{Goldreich} = \tan|\epsilon|$, offered by Goldreich (1963), relates neither to the peak work nor to the peak energy. The ambiguity in definition of Q makes this factor redundant, albeit a part of the historical tradition. Practical calculations contain the products of the Love numbers by the sines of the phase lags, $k_l\ \sin\epsilon_l$, where l is the degree of the appropriate spherical harmonic. One may though define the tidal Q by (12) and equip it with the subscript l, thus emphasising its otherness from the seismic Q (Efroimsky 2012).

3 Several basic facts from continuum mechanics

This section offers a squeezed synopsis of the basic linear solid-state mechanics. A more detailed introduction, including a glossary and examples, is offered in the Electronic Supplementary Material 1 "*Continuum mechanics. A celestial-mechanician's survival kit*".

3.1 Stationary linear deformation of isotropic incompressible media

Mechanical properties of a medium are furnished by the so-called constitutive equation or constitutive law, which interrelates the stress tensor $\mathbb{S}$ with the strain tensor $\mathbb{U}$ defined as

$$\mathbb{U} \equiv \frac{1}{2}\left[(\nabla \otimes \mathbf{u}) + (\nabla \otimes \mathbf{u})^{T}\right], \tag{15}$$

$\mathbf{u}$ being the displacement. The viscous, elastic and hereditary stresses are related to $\mathbb{U}$ via

[3] Deriving this formula in Efroimsky and Williams (2009), we inaccurately termed $E_{peak}(\chi)$ as peak energy. However our calculation of Q was carried out in understanding that $E_{peak}(\chi)$ is the peak *work*.

$$\overset{(v)}{\mathbb{S}} = \mathbb{A}\frac{\partial}{\partial t}\mathbb{U}, \quad \overset{(e)}{\mathbb{S}} = \mathbb{B}\,\mathbb{U}, \quad \overset{(h)}{\mathbb{S}} = \tilde{\mathbb{B}}\,\mathbb{U}, \tag{16}$$

where $\mathbb{A}$ is a four-dimensional matrices of viscosities, $\mathbb{B}$ is that of elastic moduli, while $\tilde{\mathbb{B}}$ is an operator-valued matrix. To furnish the value of $\sigma_{ij} = \sum_{kl} \tilde{B}_{ijkl}\, u_{kl}$ at time t, the operators $\tilde{B}_{ijkl}$ integrate the values of $u_{kl}(t')$ over the interval $t' \in (-\infty, t\ ]$.

In isotropic media, each matrix includes two terms only. The elastic stress becomes:

$$\overset{(e)}{\mathbb{S}} = \overset{(e)}{\mathbb{S}}_{volumetric} + \overset{(e)}{\mathbb{S}}_{deviatoric} = 3\,K\,\left(\frac{1}{3}\,\mathbb{I}\,\mathrm{Sp}\,\mathbb{U}\right) + 2\,\mu\,\left(\mathbb{U} - \frac{1}{3}\,\mathbb{I}\,\mathrm{Sp}\,\mathbb{U}\right), \tag{17}$$

with K and μ being the *bulk elastic modulus* and the *shear elastic modulus*, correspondingly, $\mathbb{I}$ standing for the unity matrix, and Sp denoting the trace of a matrix: $\mathrm{Sp}\,\mathbb{U} \equiv \sum_i U_{ii}$.

The hereditary stress becomes:

$$\overset{(h)}{\mathbb{S}} = \overset{(h)}{\mathbb{S}}_{volumetric} + \overset{(h)}{\mathbb{S}}_{deviatoric} = 3\,\tilde{K}\,\left(\frac{1}{3}\,\mathbb{I}\,\mathrm{Sp}\,\mathbb{U}\right) + 2\,\tilde{\mu}\,\left(\mathbb{U} - \frac{1}{3}\,\mathbb{I}\,\mathrm{Sp}\,\mathbb{U}\right), \tag{18}$$

where $\tilde{K}$ and $\tilde{\mu}$ are the *bulk-modulus operator* and the *shear-modulus* operator, accordingly.

The viscous stress acquires the form:

$$\overset{(v)}{\mathbb{S}} = \overset{(v)}{\mathbb{S}}_{volumetric} + \overset{(v)}{\mathbb{S}}_{deviatoric} = 3\,\zeta\,\frac{\partial}{\partial t}\,\left(\frac{1}{3}\,\mathbb{I}\,\mathrm{Sp}\,\mathbb{U}\right) + 2\,\eta\,\frac{\partial}{\partial t}\,\left(\mathbb{U} - \frac{1}{3}\,\mathbb{I}\,\mathrm{Sp}\,\mathbb{U}\right), \tag{19}$$

the quantities ζ and η being termed as the *bulk* and *shear viscosity*, correspondingly.

The term $\frac{1}{3}\,\mathbb{I}\,\mathrm{Sp}\,\mathbb{U}$ is called the *volumetric* part of the strain, while $\mathbb{U} - \frac{1}{3}\,\mathbb{I}\,\mathrm{Sp}\,\mathbb{U}$ is called the *deviatoric* part. Accordingly, in expressions (17–19) for the stresses, the pure-trace terms are called *volumetric*, the other term being named *deviatoric*.

If an isotropic medium is also incompressible, the relative change of the volume, $\mathrm{Sp}\,\mathbb{U} = 0$, vanishes; so the volumetric parts of the strain and stresses become nil. The incompressibility assumption may be applicable both to crusty and to low-porous icy bodies. At least for Iapetus, the low-porosity assumption is likely to be correct (Castillo-Rogez et al. 2011).

3.2 Evolving stresses and strains: basic notations

In the general case, loading varies in time, so one has to deal with the stress and strain tensors as functions of time. However, treatment of viscoelasticity turns out to be simpler in the frequency domain, i.e., in the language of complex rigidity and complex compliance. To this end, the stress $\sigma_{\gamma\nu}$ and strain $u_{\gamma\nu}$ in a linear medium can be Fourier-expanded as

$$\sigma_{\gamma\nu}(t) = \sum_{n=0}^{\infty} \sigma_{\gamma\nu}(\chi_n)\,\cos\left[\chi_n t + \varphi_\sigma(\chi_n)\right] = \sum_{n=0}^{\infty} \mathcal{R}e\left[\bar{\sigma}_{\gamma\nu}(\chi_n)\quad \exp\left(i\chi_n t\right)\right], \tag{20}$$

$$u_{\gamma\nu}(t) = \sum_{n=0}^{\infty} u_{\gamma\nu}(\chi_n)\,\cos\left[\chi_n t + \varphi_u(\chi_n)\right] = \sum_{n=0}^{\infty} \mathcal{R}e\left[\bar{u}_{\gamma\nu}(\chi_n)\quad \exp\left(i\chi_n t\right)\right], \tag{21}$$

where the complex amplitudes are:

$$\bar{\sigma}_{\gamma\nu}(\chi) = \sigma_{\gamma\nu}(\chi)\exp\left[i\varphi_\sigma(\chi)\right], \quad \bar{u}_{\gamma\nu}(\chi) = u_{\gamma\nu}(\chi)\exp\left[i\varphi_u(\chi)\right], \tag{22}$$

while the initial phases $\varphi_\sigma(\chi)$ and $\varphi_u(\chi)$ are chosen in a manner that sets the real amplitudes $\sigma_{\gamma\nu}(\chi_n)$ and $u_{\gamma\nu}(\chi_n)$ non-negative.

We wrote the above expansions as sums over a discrete spectrum, as the spectrum generated by tides is discrete. Generally, the sums can be replaced with integrals over frequency:

$$\sigma_{\gamma\nu}(t) = \int_0^\infty \bar{\sigma}_{\gamma\nu}(\chi)\, e^{i\chi t}\, d\chi \qquad \text{and} \qquad u_{\gamma\nu}(t) = \int_0^\infty \bar{u}_{\gamma\nu}(\chi)\, e^{i\chi t}\, d\chi . \tag{23}$$

Whenever necessary, the frequency is set to approach the real axis from below: $\mathcal{I}m(\chi) \to 0-$.

3.3 Should we consider positive frequencies only?

At first glance, the above question appears pointless, because in classical physics a negative frequency is a mere abstraction. Mathematically, a full Fourier decomposition of a *real* field can always be reduced to a decomposition over positive frequencies only.

For example, the full Fourier integral for the stress can be written as

$$\sigma_{\gamma\nu}(t) = \int_{-\infty}^{\infty} \bar{s}_{\gamma\nu}(\omega)\, e^{i\omega t} d\omega = \int_0^\infty \left[\bar{s}_{\gamma\nu}(\chi)\, e^{i\chi t} + \bar{s}_{\gamma\nu}(-\chi)\, e^{-i\chi t}\right] d\chi , \tag{24}$$

where we define $\chi \equiv |\omega|$. Denoting complex conjugation with asterisk, we write:

$$\sigma^*_{\gamma\nu}(t) = \int_0^\infty \left[\bar{s}^*_{\gamma\nu}(-\chi)\, e^{i\chi t} + \bar{s}^*_{\gamma\nu}(\chi)\, e^{-i\chi t}\right] d\chi . \tag{25}$$

The stress is real: $\sigma^*_{\gamma\nu}(t) = \sigma_{\gamma\nu}(t)$. Equating the right-hand sides of (24) and (25), we obtain

$$\bar{s}_{\gamma\nu}(-\chi) = \bar{s}^*_{\gamma\nu}(\chi), \tag{26}$$

whence

$$\sigma_{\gamma\nu}(t) = \int_0^\infty \left[\bar{s}_{\gamma\nu}(\chi)\, e^{i\chi t} + \bar{s}^*_{\gamma\nu}(\chi)\, e^{-i\chi t}\right] d\chi = \mathcal{R}e \int_0^\infty 2\, \bar{s}_{\gamma\nu}(\chi)\, e^{i\chi t}\, d\chi . \tag{27}$$

This leads us to (23), if we set

$$\bar{\sigma}_{\gamma\nu}(\chi) = 2\, \bar{s}_{\gamma\nu}(\chi). \tag{28}$$

Our switch from $\sigma_{\gamma\nu}(t) = \int_{-\infty}^{\infty} \bar{s}_{\gamma\nu}(\omega)\, e^{i\omega t}\, d\omega$ to the expansion $\sigma_{\gamma\nu}(t) = \int_0^\infty \bar{\sigma}_{\gamma\nu}(\omega)\, e^{i\chi t}\, d\chi$ makes things simpler, but the simplification comes at a cost, as we shall see in a second.

Recall that the tide can be expanded over the modes

$$\omega_{lmpq} \equiv (l-2p)\,\dot{\omega} + (l-2p+q)\,\dot{\mathcal{M}} + m\,(\dot{\Omega} - \dot{\theta}) \approx (l-2p+q)\,n - m\,\dot{\theta}, \tag{29}$$

each of which can assume positive or negative values, or be zero. Here l, m, p, q are some integers, θ is the primary's sidereal angle, $\dot{\theta}$ is its spin rate, while ω, Ω, $\mathcal{M}$ and n are the secondary's periapse, node, mean anomaly, and mean motion. The tidal frequencies, at which the medium gets loaded, are given by the modes' absolute values: $\chi_{lmpq} \equiv |\,\omega_{lmpq}\,|$.

The positively-defined forcing frequency χ_{lmpq} is the actual physical frequency at which the $lmpq$ term in the expansion for the tidal potential (or stress or strain) oscillates.

The motivation for keeping also the modes ω_{lmpq} is subtle: it depends upon the *sign* of ω_{lmpq} whether the $lmpq$ component of the tide lags or advances. Specifically, the phase lag

between the $lmpq$ component of the perturbed primary's potential U and the $lmpq$ component of the tide-raising potential W generated by the secondary is given by

$$\epsilon_{lmpq} = \omega_{lmpq}\,\Delta t_{lmpq} = |\,\omega_{lmpq}\,|\,\Delta t_{lmpq}\ \mathrm{sgn}\,\omega_{lmpq} = \chi_{lmpq}\,\Delta t_{lmpq}\ \mathrm{sgn}\,\omega_{lmpq}\,, \tag{30}$$

where the time lag Δt_{lmpq} is always positive.

While the lag between the applied stress and resulting strain in a sample of a medium is always positive, the tidal lag can be positive or negative. This in no way implies violation of causality (the time lag Δt_{lmpq} is always positive). Rather, it reflects the directional difference between the planetocentric positions of the tide-raising body and the resulting bulge. For example, the principal tide, $lmpq = 2200$, stays behind (has a positive phase lag ϵ_{2200}) when the secondary is below the synchronous orbit, and advances (has a negative phase lag ϵ_{2200}) when the secondary is at a higher orbit. To summarise, decomposition of a tide over both positive and negative modes ω_{lmpq} (and not just over the positive frequencies χ_{lmpq}) does have a physical meaning, as the sign of a mode ω_{lmpq} carries physical information.

Thus we arrive at the following conclusions:

1. As the fields emerging in the tidal theory—the tidal potential, stress, and strain—are all real, their expansions in the frequency domain may, in principle, be written down using the positive frequencies χ only.
2. The tidal potential contains components corresponding to the tidal modes ω_{lmpq} of both the positive and negative signs. While the $lmpq$ components of the potential, stress, and strain oscillate at the positive frequencies $\chi_{lmpq} = |\omega_{lmpq}|$, the sign of ω_{lmpq} points whether the lagging of the $lmpq$ component of the bulge is positive or negative (falling behind or advancing). This sign enters the expression for the appropriate component of the torque or force. Hence a consistent tidal theory should employ expansions over both positive and negative tidal modes ω_{lmpq} and not just over the positive χ_{lmpq}.
3. In order to rewrite the tidal theory in terms of the positively-defined frequencies χ_{lmpq} only, one must inserts "by hand" the extra multipliers

$$\mathrm{sgn}\,\omega_{lmpq} = \mathrm{sgn}\left[(l-2p+q)\,n - m\,\dot{\theta}\right] \tag{31}$$

   into the expressions for the $lmpq$ components of the tidal torque and force.
4. One can employ a rheological law (constitutive equation interconnecting the strain and stress) and a Navier–Stokes equation (the second law of Newton for an element of the medium) to calculate the phase lag ϵ_{lmpq} of the primary's potential U_{lmpq} relative to the potential W_{lmpq} generated by the secondary. If both these equations are expanded, in the frequency domain, via positively-defined forcing frequencies χ_{lmpq} only, the resulting phase lag, too, will emerge as a function of χ_{lmpq}:

$$\epsilon_{lmpq} = \epsilon_l\,(\chi_{lmpq}). \tag{32}$$

   Within this treatment, one has to equip the lag, "by hand", with the sign factor (31). On crossing of an $lmpq$ resonance, the factor will change its sign. Accordingly, the $lmpq$ term of the tidal torque (or force) will change its sign too.

The lag (32) is the negative argument of the complex Love number $\bar{k}_l\,(\chi_{lmpq})$. Solution of the constitutive and Navier–Stokes equations renders the complex Love numbers, from which one can calculate the lags. Hence the above item [4] may be rephrased in the following manner:

4′. Under the convention that $U_{lmpq} = U(\chi_{lmpq})$ and $W_{lmpq} = W(\chi_{lmpq})$, we have:

$$U_{lmpq} = \bar{k}_l\,(\chi_{lmpq})\,W_{lmpq} \quad \text{when} \quad \omega_{lmpq} > 0, \quad \text{i.e., when} \quad \omega_{lmpq} = \chi_{lmpq}\,, \tag{33a}$$

$$U_{lmpq} = \bar{k}_l^*(\chi_{lmpq})\, W_{lmpq} \quad \text{when} \quad \omega_{lmpq} < 0, \quad \text{i.e., when} \quad \omega_{lmpq} = -\chi_{lmpq}, \tag{33b}$$

the asterisk denoting the complex conjugation.

This ugly convention, a switch from $\bar{k}_l$ to $\bar{k}_l^*$, is the price we pay for employing only the positive frequencies in our expansions, when solving the constitutive and Navier–Stokes equations, to find the Love number. In other words, this is a price for our pretending that W_{lmpq} and U_{lmpq} are functions of χ_{lmpq}—whereas in reality they are functions of ω_{lmpq}.

Alternative to this would be expanding the stress, strain, and the potentials over the positive and negative modes ω_{lmpq}, with the negative frequencies showing up in the equations. With the convention that $U_{lmpq} = U(\omega_{lmpq})$ and $W_{lmpq} = W(\omega_{lmpq})$, we would have

$$U_{lmpq} = \bar{k}_l(\omega_{lmpq})\, W_{lmpq}, \quad \text{for all} \quad \omega_{lmpq}. \tag{34}$$

3.4 The complex rigidity and compliance: stress–strain relaxation

The stress cannot be obtained by means of an integral operator that would map the past history of the strain, $\mathbb{U}(t')$ over $t' \in (-\infty,\ t]$, to the value of $\mathbb{S}$ at time t. The insufficiency of such an operator stems from the presence of a time-derivative on the right-hand side of the viscous part of the stress, given by the first expression in (16). Exceptional are the cases of no viscosity (e.g., a purely elastic material).

On the other hand, we expect on physical grounds that the operator $\hat{J}$ inverse to $\hat{\mu}$ *is* an integral operator. In other words, we assume that the current value of strain depends only on the present and past values taken by the stress and not on the current *rate* of change of the stress. The assumption works for weak deformations, i.e., insofar as no plasticity shows up. So we assume that the operator $\hat{J}$ is just an integral operator.

Since the forced medium "remembers" the history of loading, the strain at time t must be a sum of small installments $\frac{1}{2}\, J(t - t')\, d\sigma_{\gamma\nu}(t')$, each of which stems from a small change $d\sigma_{\gamma\nu}(t - \tau)$ of the stress at an earlier time $t' < t$. The entire history of the past loading results, at the time t, in a total strain $u_{\gamma\nu}(t)$ rendered by an integral operator $\hat{J}(t)$ acting on the entire function $\sigma_{\gamma\nu}(t')$ and not on its particular value (Karato 2008):

$$2\, u_{\gamma\nu}(t) = \hat{J}(t)\, \sigma_{\gamma\nu} = \int_0^\infty J(\tau)\ \stackrel{\bullet}{\sigma}_{\gamma\nu}(t - \tau)\, d\tau = \int_{-\infty}^t J(t - t')\ \stackrel{\bullet}{\sigma}_{\gamma\nu}(t')\, dt', \tag{35}$$

where t' is some earlier time ($t' < t$), overdot denotes d/dt', while the "age variable" $\tau = t - t'$ is reckoned from the current moment t and is aimed back into the past. The so-defined integral operator $\hat{J}(t)$ is called the *compliance operator*, while its kernel $J(t - t')$ goes under the name of the *compliance function* or the *creep-response function*.

Integrating (35) by parts, we recast the compliance operator into the form of

$$2\, u_{\gamma\nu}(t) = \hat{J}(t)\, \sigma_{\gamma\nu} = J(0)\, \sigma_{\gamma\nu}(t) - J(\infty)\, \sigma_{\gamma\nu}(-\infty) + \int_0^\infty \stackrel{\bullet}{J}(\tau)\, \sigma_{\gamma\nu}(t - \tau) d\tau \tag{36a}$$

$$= J(0)\, \sigma_{\gamma\nu}(t) - J(\infty)\, \sigma_{\gamma\nu}(-\infty) + \int_{-\infty}^t \stackrel{\bullet}{J}(t - t')\, \sigma_{\gamma\nu}(t')\, dt'. \tag{36b}$$

The quantity $J(\infty)$ is the *relaxed compliance*. Being the asymptotic value of $J(t-t')$ at $t-t' \rightarrow \infty$, this parameter corresponds to the strain after complete relaxation. The load in the infinite past may be assumed zero, and the term $-J(\infty)\,\sigma_{\gamma\nu}(-\infty)$ may be dropped.

The second important quantity emerging in (36) is the *unrelaxed compliance* $J(0)$, which is the value of the compliance function $J(t-t')$ at $t-t'=0$. This parameter describes the instantaneous reaction to stressing, and thus defines the *elastic* part of the deformation (the rest of the deformation being viscous and hereditary). Thus the term containing the unrelaxed compliance $J(0)$ should be kept. The term, though, can be absorbed into the integral if we agree that the elastic contribution enters the compliance function not as[4]

$$J(t-t') = J(0) + \text{viscous and hereditary terms}, \tag{37}$$

but as

$$J(t-t') = J(0)\,\Theta(t-t') + \text{viscous and hereditary terms}, \tag{38}$$

the Heaviside step-function $\Theta(t-t')$ being unity for $t-t' \geq 0$ and zero for $t-t' < 0$. As the derivative of the step-function is the delta-function $\delta(t-t')$, we can write (36b) as

$$2u_{\gamma\nu}(t) = \int_{-\infty}^{t} \dot{J}\,(t-t')\,\sigma_{\gamma\nu}(t')dt', \quad \text{with} \quad J(t-t') \quad \text{containing} \quad J(0)\,\Theta(t-t'). \tag{39}$$

Formulae (35–36) and (39) are but different expressions for the compliance operator $\hat{J}$ acting as

$$2\,u_{\gamma\nu} = \hat{J}\,\sigma_{\gamma\nu}. \tag{40}$$

Inverse to the compliance operator is the rigidity operator $\hat{\mu}$ defined through

$$\sigma_{\gamma\nu} = 2\,\hat{\mu}\,u_{\gamma\nu}. \tag{41}$$

Generally, $\hat{\mu}$ is not an integral operator, but is an integro-differential operator. So it cannot take the form of $\sigma_{\gamma\nu}(t) = 2\int_{-\infty}^{t}\dot{\mu}(t-t')\,u_{\gamma\nu}(t')\,dt'$. However it can be written as

$$\sigma_{\gamma\nu}(t) = 2\int_{-\infty}^{t} \mu(t-t')\,\dot{u}_{\gamma\nu}(t')\,dt', \tag{42}$$

if we permit the kernel $\mu(t-t')$ to contain a term $\eta\,\delta(t-t')$, where $\delta(t-t')$ is the delta-function. After integration, this term will furnish the viscous part of the stress, $2\,\eta\,\dot{u}_{\gamma\nu}$.

The kernel $\mu(t-t')$ is termed the *stress-relaxation function*. Its time-independent part is $\mu(0)\,\Theta(t-t')$, where the *unrelaxed rigidity* $\mu(0)$ is inverse to the unrelaxed compliance $J(0)$ and describes the elastic part of deformation. Each term in $\mu(t-t')$, which neither is a constant nor contains a delta-function, is responsible for hereditary reaction.

[4] Expressing the stress through the strain, we encountered three possibilities: the elastic stress was simply proportional to the strain, the viscous stress was proportional to the time-derivative of the strain, while the hereditary stress was expressed by an integral operator $\tilde{\mu}$. However, when we express the strain through the stress, we place the viscosity into the integral operator, so the viscous reaction also looks hereditary. It is our convention, though, to apply the term *hereditary* to delayed reactions *other than purely viscous*.

3.5 Stress–strain relaxation in the frequency domain

The complex compliance $\bar{J}(\chi)$ and the complex rigidity $\bar{\mu}(\chi)$ are, by definition, the Fourier images **not** of the $J(\tau)$ and $\mu(\tau)$ functions, but of their time-derivatives:[5]

$$\int_0^{\infty} \bar{J}(\chi)\, e^{i\chi\tau} d\chi = \dot{J}\,(\tau), \quad \text{where} \quad \bar{J}(\chi) = \int_0^{\infty} \dot{J}\,(\tau)\, e^{-i\chi\tau}\, d\tau. \tag{43}$$

and

$$\int_0^{\infty} \bar{\mu}(\chi)\, e^{i\chi\tau} d\chi = \dot{\mu}(\tau), \quad \text{where} \quad \bar{\mu}(\chi) = \int_0^{\infty} \dot{\mu}(\tau)\, e^{-i\chi\tau}\, d\tau, \tag{44}$$

the integrations over τ spanning the interval $[0, \infty)$, as both kernels are nil for $\tau < 0$ anyway. In (43) and (44), we made use of the fact (explained in Sect. 3.3) that, when expanding real fields, it is sufficient to use only positive frequencies.

Formula (35), in combination with the Fourier expansions (23) and with (43), furnishes:

$$2 \int_0^{\infty} \bar{u}_{\gamma\nu}(\chi)\, e^{i\chi t}\, d\chi = \int_0^{\infty} \bar{\sigma}_{\mu\nu}(\chi)\, \bar{J}(\chi)\, e^{i\chi t}\, d\chi, \tag{45}$$

which leads us to:

$$2\, \bar{u}_{\gamma\nu}(\chi) = \bar{J}(\chi)\, \bar{\sigma}_{\gamma\nu}(\chi). \tag{46}$$

Similarly, insertion of (23) into (42) leads to the relation

$$\bar{\sigma}_{\gamma\nu}(\chi) = 2\, \bar{\mu}(\chi)\, \bar{u}_{\gamma\nu}(\chi), \tag{47}$$

comparison whereof with (46) immediately entails:

$$\bar{J}(\chi)\, \bar{\mu}(\chi) = 1. \tag{48}$$

Writing down the complex rigidity and compliance as

$$\bar{\mu}(\chi) = |\bar{\mu}(\chi)|\ \exp\left[i\,\delta(\chi)\right] \tag{49}$$

and

$$\bar{J}(\chi) = |\bar{J}(\chi)|\ \exp\left[-i\,\delta(\chi)\right], \tag{50}$$

we split (48) into two expressions:

$$|\bar{J}(\chi)| = \frac{1}{|\bar{\mu}(\chi)|} \tag{51}$$

and

$$\varphi_u(\chi) = \varphi_\sigma(\chi) - \delta(\chi). \tag{52}$$

From the latter, we see that the angle $\delta(\chi)$ is a measure of lagging of a strain harmonic mode relative to the appropriate harmonic mode of the stress. It ensues from (49–50) that

$$\tan\delta(\chi)\ \equiv -\frac{\mathcal{I}m\left[\bar{J}(\chi)\right]}{\mathcal{R}e\left[\bar{J}(\chi)\right]} = \frac{\mathcal{I}m\left[\bar{\mu}(\chi)\right]}{\mathcal{R}e\left[\bar{\mu}(\chi)\right]}. \tag{53}$$

[5] Recall that it is the time-derivative of $J(\tau)$ that is the kernel of the integral operator (39). Hence, to arrive at (46), we have to define $\bar{J}(\chi)$ as the Fourier image of $\dot{J}\,(\tau)$.

4 Complex Love numbers

The developments presented in this section will rest on a theorem, which is known as the *correspondence principle* or the *elastic-viscoelastic analogy*, and which applies to linear deformations in the absence of nonconservative (inertial) forces. While the literature attributes the authorship of the theorem to different scholars, its true pioneer was Sir George Darwin (1879). One of the corollaries ensuing from this theorem is that, in the frequency domain, the complex Love numbers are expressed via the complex rigidity or compliance in the same way as the static Love numbers are expressed via the relaxed rigidity or compliance.

As was pointed out by Biot (1954, 1958), the theorem is inapplicable to non-potential forces. Hence the said corollary fails in the case of librating bodies, because of the presence of the inertial force[6] $-\dot{\boldsymbol{\omega}}\times\mathbf{r}\,\rho$, where ρ is the density and $\boldsymbol{\omega}$ is the libration angular velocity. So the expression (3) for the Love numbers, generally, cannot be employed for librating bodies.

Section 4.1 below explains the transition from the stationary Love numbers to their dynamical counterparts, the Love operators. We present this formalism in the frequency domain, in the spirit of Zahn (1966) who pioneered this approach in application to a purely viscous medium. Section 4.2 addresses the negative tidal modes emerging in the Darwin–Kaula expansion for tides. Employing the correspondence principle, in Sect. 4.3 we write down the expressions for the factors $|\bar{k}_l(\chi)|\,\sin\epsilon_l(\chi) = -\mathcal{I}m[\bar{k}_l(\chi)]$ emerging in the expansion for tides. Technical details of this derivation are discussed in Sects. 4.4–4.5.

For more on the correspondence principle and its applicability to Phobos see Appendix B.

4.1 From the Love numbers to the Love operators

A homogeneous incompressible primary, when perturbed by a static secondary, yields its form and, consequently, has its potential changed. The l^{th} spherical harmonic $U_l(\mathbf{r})$ of the resulting increment of the primary's exterior potential is related to the l^{th} spherical harmonic $W_l(\mathbf{R},\,\mathbf{r})$ of the perturbing exterior potential through (2).

Under evolving disturbances, the Love numbers become operators mapping the function $W_l(\mathbf{R},\ \mathbf{r}^*,\ t\,')$, with $t\,'\in(-\infty,\ t]$, onto a value of U_l at time t:

$$U_l(\mathbf{r},\ t) = \left(\frac{R}{r}\right)^{l+1}\ \hat{k}_l(t)\ W_l(\mathbf{R},\ \mathbf{r}^*,\ t\,'). \tag{54}$$

Being linear for weak forcing, the operators must read:

$$U_l(\mathbf{r},\ t) = \left(\frac{R}{r}\right)^{l+1}\int_{\tau=0}^{\tau=\infty} k_l(\tau)\ \dot{W}_l\ (\mathbf{R},\mathbf{r}^*,\ t-\tau)\,d\tau$$

$$= \left(\frac{R}{r}\right)^{l+1}\int_{t\,'=-\infty}^{t\,'=t} k_l(t-t\,')\ \dot{W}_l\ (\mathbf{R},\mathbf{r}^*,\ t\,')\,dt\,' \tag{55a}$$

or, after integration by parts:

$$U_l(\mathbf{r},\ t) = \left(\frac{R}{r}\right)^{l+1}[k_l(0)W(t) - k_l(\infty)W(-\infty)] + \left(\frac{R}{r}\right)^{l+1}\int_0^{\infty}\dot{k}_l(\tau)\ W_l(\mathbf{R},\ \mathbf{r}^*,\ t-\tau)d\tau \tag{55b}$$

[6] The centripetal term is potential and causes no troubles, except for the need for a degree-0 Love number.

$$= \left(\frac{R}{r}\right)^{l+1} [k_l(0)W(t) - k_l(\infty)W(-\infty)] + \left(\frac{R}{r}\right)^{l+1} \int_{-\infty}^{t} \dot{k}_l(t-t')\, W_l(\mathbf{R}, \mathbf{r}^*, t')dt' \tag{55c}$$

$$= -\left(\frac{R}{r}\right)^{l+1} k_l(\infty)W(-\infty) + \left(\frac{R}{r}\right)^{l+1} \int_{-\infty}^{t} \frac{d}{dt}\Big[\, k_l(t-t') - k_l(0) + k_l(0)\Theta(t-t')\Big]\, W_l(\mathbf{R}, \mathbf{r}^*, t')\, dt'. \tag{55d}$$

Just as in the case of the compliance operator (35–36), in expressions (55) we obtain the terms $k_l(0)W(t)$ and $-k_l(\infty)W(-\infty)$. Of the latter term, we can get rid by setting $W(-\infty)$ nil, while the former term may be incorporated into the kernel in exactly the same way as in (37–39). Thus, dropping the unphysical term with $W(-\infty)$, and inserting the elastic term into the Love number not as $k_l(0)$ but as $k_l(0)\,\Theta(t-t')$, we simplify (55d) to

$$U_l(\mathbf{r},\ t) = \left(\frac{R}{r}\right)^{l+1} \int_{-\infty}^{t} \dot{k}_l(t-t')\, W_l(\mathbf{R},\ \mathbf{r}^*,\ t')\, dt', \tag{56}$$

with $k_l(t-t')$ now including, as its part, $k_l(0)\,\Theta(t-t')$ instead of $k_l(0)$.

Were the body perfectly elastic, $k_l(t-t')$ would consist of the instantaneous-reaction term $k_l(0)\,\Theta(t-t')$ *only*. Accordingly, the time-derivative of k_l would be: $\dot{k}_l(t-t') = k_l\,\delta(t-t')$ where $k_l \equiv k_l(0)$, so expressions (55–56) would coincide with (2).

Similarly to introducing the complex compliance, one can define the complex Love numbers as Fourier transforms of $\dot{k}_l\ (\tau)$:

$$\int_0^{\infty} \bar{k}_l(\chi) e^{i\chi\tau} d\chi = \dot{k}_l\ (\tau), \tag{57}$$

the overdot standing for $d/d\tau$. Churkin (1998) suggested to term the time-derivatives $\dot{k}_l\ (t)$ as the *Love functions*.[7] Inversion of (57) trivially yields:

$$\bar{k}_l(\chi) = \int_0^{\infty} \dot{k}_l(\tau)\quad e^{-i\chi\tau}\, d\tau = k_l(0) + i\,\chi \int_0^{\infty} \Big[k_l(\tau) - k_l(0)\,\Theta(\tau)\Big]\ e^{-i\chi\tau}\, d\tau, \tag{58}$$

where we integrated only from 0 because the future disturbance contributes nothing to the present distortion, so $k_l(\tau)$ vanishes at $\tau < 0$. Recall that the time τ denotes the difference $t - t'$. So τ is reckoned from the present moment t and is directed back into the past.

Defining in the standard manner the Fourier components $\bar{U}_l(\chi)$ and $\bar{W}_l(\chi)$ of functions $U_l(t)$ and $W_l(t)$, we write (55) in the frequency domain:

$$\bar{U}_l(\chi) = \left(\frac{R}{r}\right)^{l+1} \bar{k}_l(\chi)\quad \bar{W}_l(\chi), \tag{59}$$

where we denote the frequency simply by χ instead of the awkward χ_{lmpq}. To employ (59) in the tidal theory, one has to know the frequency-dependencies $\bar{k}_l(\chi)$.

[7] Churkin (1998) used functions $k_l(t)$ which were, due to a difference in notations, the same as our $\dot{k}_l\ (\tau)$.

4.2 The positive forcing frequencies $\chi \equiv |\omega|$ versus the positive and negative tidal modes ω

It should be remembered that, by relying on formula (59), we place ourselves on thin ice, because the similarity of this formula to (46) and (47) is deceptive.

In (46) and (47), it was legitimate to limit our expansions of the stress and the strain to positive frequencies χ only. Had we carried out those expansions over both positive and negative frequencies ω, we would have obtained, instead of (46) and (47), similar expressions

$$2\,\bar{u}_{\gamma\nu}(\omega) = \bar{J}(\omega)\,\bar{\sigma}_{\gamma\nu}(\omega) \qquad \text{and} \qquad \bar{\sigma}_{\gamma\nu}(\omega) = 2\,\bar{\mu}(\omega)\,\bar{u}_{\gamma\nu}(\omega). \tag{60}$$

For positive ω, these would simply coincide with (46) and (47), had we renamed ω as χ. For negative $\omega = -\chi$, the resulting expressions would read as

$$2\,\bar{u}_{\gamma\nu}(-\chi) = \bar{J}(-\chi)\,\bar{\sigma}_{\gamma\nu}(-\chi) \qquad \text{and} \qquad \bar{\sigma}_{\gamma\nu}(-\chi) = 2\,\bar{\mu}(-\chi)\,\bar{u}_{\gamma\nu}(-\chi), \tag{61}$$

where we stick to the agreement that χ always stands for a positive quantity. In accordance with (26), complex conjugation of (61) would then return us to (60).

Physically, the negative-frequency components of the stress or strain are nonexistent. If brought into consideration, they are obliged to obey (26) and, thus, should play no role, except for a harmless renormalisation of the Fourier components in (28).

When we say that the physically measurable stress $\sigma_{\gamma\nu}(t)$ is equal to $\sum \mathcal{R}e\left[\bar{\sigma}_{\gamma\nu}(\chi)\,e^{i\chi t}\right]$, it is unimportant to us whether the χ-contribution in $\sigma_{\gamma\nu}(t)$ comes from the term $\bar{\sigma}_{\gamma\nu}(\chi)\,e^{i\chi t}$ only, or also from the term $\bar{\sigma}_{\gamma\nu}(-\chi)\,e^{i(-\chi)t}$. Indeed, the real part of the latter is a clone of the real part of the former (and it is only the former term that is physical). However, things remain that simple only for the stress and the strain.

As we emphasised in Sect. 3.3, the situation with the potentials is drastically different. While the potential $U(t)$ is still equal to $\sum \mathcal{R}e\left[\bar{U}(\chi)\,e^{i\chi t}\right]$, it is now important to distinguish whether the χ-contribution into $U(t)$ comes from the term $\bar{U}_{\gamma\nu}(\chi)\,e^{i\chi t}$ or from the term $\bar{U}(-\chi)\,e^{i(-\chi)t}$, or from both. Although the negative mode $(-\chi)$ would bring the same input as the positive mode χ, the inputs will contribute differently into the tidal torque. As can be seen from (109), the secular part of the torque is proportional to $\sin\epsilon_l$, where $\epsilon_l \equiv \omega_{lmpq}\,\Delta t_{lmpq}$, with the time lag Δt_{lmpq} being positively defined—see formula (105). So the secular part of the torque explicitly contains the sign of the tidal mode ω_{lmpq}.

Thus, as explained in Sect. 3.3, a more accurate form of formula (59) should be:

$$\bar{U}_l(\omega) = \bar{k}_l(\omega)\;\bar{W}_l(\omega), \tag{62}$$

where ω can be of any sign.

If however we pretend that the potentials depend on the physical frequency $\chi = |\omega|$ only, i.e., if we always write $U(\omega)$ as $U(\chi)$, then (59) must be written as:

$$\bar{U}_l(\chi) = \bar{k}_l(\chi)\;\bar{W}_l(\chi), \quad \text{when} \quad \chi = |\omega| \quad \text{for} \quad \omega > 0, \tag{63a}$$

and

$$\bar{U}_l(\chi) = \bar{k}_l^{*}(\chi)\;\bar{W}_l(\chi), \quad \text{when} \quad \chi = |\omega| \quad \text{for} \quad \omega < 0. \tag{63b}$$

Unless we keep this detail in mind, we shall get a wrong sign for the $lmpq$ component of the torque after the despinning secondary crosses the appropriate commensurability. (We shall, of course, be able to mend this by simply inserting the sign sgn ω_{lmpq} by hand.)

4.3 The complex Love number as a function of the complex compliance

While the static Love numbers depend on the static rigidity μ via (3), it is not readily apparent that the same relation link $\bar{k}_l(\chi)$ to $\bar{\mu}(\chi)$. The *correspondence principle* discussed in Appendix B tells us that in many situations the viscoelastic operational moduli $\bar{\mu}(\chi)$, $\bar{J}(\chi)$ obey the same algebraic relations as the elastic parameters μ, J. This is why in these situations the Fourier or Laplace transform of our viscoelastic equations mimic (144), except that the functions acquire overbars: $\bar{\sigma}_{\gamma\nu} = 2\,\bar{\mu}\,\bar{u}_{\gamma\nu}$, etc. So the solution is $\bar{U}_l = \bar{k}_l\,\bar{W}_l$, with $\bar{k}_l$ retaining the same dependence on ρ, R and $\bar{\mu}$ as in (3), except that now μ have an overbar:

$$\bar{k}_l(\chi) = \frac{3}{2\,(l-1)}\;\frac{1}{1+\dfrac{(2\,l^2+4\,l+3)\,\bar{\mu}(\chi)}{l\,\mathrm{g}\,\rho\,R}} = \frac{3}{2\,(l-1)}\;\frac{1}{1+A_l\;\bar{\mu}(\chi)/\mu}$$
$$= \frac{3}{2\,(l-1)}\;\frac{1}{1+A_l\;J/\bar{J}(\chi)} = \frac{3}{2\,(l-1)}\;\frac{\bar{J}(\chi)}{\bar{J}(\chi)+A_l\;J} \tag{64}$$

Here the coefficients A_l are defined via the unrelaxed quantities $\mu = \mu(0) = 1/J = 1/J(0)$ in the same way as the static A_l were introduced via the static (relaxed) $\mu = \mu(\infty) = 1/J(\infty)$ in (3).

The moral of the story is that at low frequencies each $\bar{k}_l$ depends on $\bar{\mu}$ or $\bar{J}$ in the same way as the static k_l depends on the static μ or J. This happens because at low frequencies we drop the acceleration term in the equation of motion (147b), so (147b) looks like (144a).

Representing a complex Love number as

$$\bar{k}_l(\chi) = \mathcal{R}e\left[\bar{k}_l(\chi)\right] + i\;\mathcal{I}m\left[\bar{k}_l(\chi)\right] = |\bar{k}_l(\chi)|\;e^{-i\epsilon_l(\chi)} \tag{65}$$

we can write for the phase lag $\epsilon_l(\chi)$:

$$\tan\epsilon_l(\chi) \equiv -\frac{\mathcal{I}m\left[\bar{k}_l(\chi)\right]}{\mathcal{R}e\left[\bar{k}_l(\chi)\right]} \tag{66}$$

or, equivalently:

$$|\bar{k}_l(\chi)|\;\sin\epsilon_l(\chi) = -\mathcal{I}m\left[\bar{k}_l(\chi)\right]. \tag{67}$$

The products $|\bar{k}_l(\chi)|\;\sin\epsilon_l(\chi)$ standing on the left-hand side in (67) emerge also in the Fourier series for the tidal potential. So it is these products, not k_l/Q, that enter the expansions for forces, torques and the damping rate. They link the body's rheology with its spin history: from $\bar{J}(\chi)$ to $\bar{k}_l(\chi)$ to $|\bar{k}_l(\chi)|\;\sin\epsilon(\chi)$, the latter being employed in the theory of tides.

Through simple algebra, expressions (64) entail:

$$|\bar{k}_l(\chi)|\;\sin\epsilon_l(\chi) = -\mathcal{I}m\left[\bar{k}_l(\chi)\right] = \frac{3}{2\,(l-1)}\;\frac{-A_l\;J\;\mathcal{I}m\left[\bar{J}(\chi)\right]}{\left(\mathcal{R}e\left[\bar{J}(\chi)\right]+A_lJ\right)^2+\left(\mathcal{I}m\left[\bar{J}(\chi)\right]\right)^2}. \tag{68}$$

As we know from Sects. 3.3 and 4.2, formulae (66–68) should be used with care. Since in reality $\bar{U}$ and $\bar{k}_l$ are functions not of χ but of ω, then formulae (68) should be equipped with multipliers sgn ω_{lmpq}, when plugged into the expression for the $lmpq$ component of the tidal force or torque. This prescription is equivalent to (63).

4.4 Should we write $\bar{k}_{lmpq}$ and ϵ_{lmpq}, or would $\bar{k}_l$ and ϵ_l be enough?

In the preceding subsection, the static relation (2) was generalised to evolving settings as

$$U_{lmpq}(\mathbf{r},\ t) = \left(\frac{R}{r}\right)^{l+1} \hat{k}_l(t)\, W_{lmpq}(\mathbf{R},\ \mathbf{r}^*,\ t\,'), \tag{69}$$

where $lmpq$ is a quadruple of integers employed to number a Fourier mode in the Darwin–Kaula expansion (96) of the tide, while $U_{lmpq}(\mathbf{r},\ t)$ and $W_{lmpq}(\mathbf{R},\ \mathbf{r}^*,\ t\,')$ are the harmonics containing $\cos(\chi_{lmpq} t - \epsilon_{lmpq})$ and $\cos(\chi_{lmpq} t\,')$ correspondingly.

One might be tempted to generalise (2) even further to

$$U_{lmpq}(\mathbf{r},\ t) = \left(\frac{R}{r}\right)^{l+1} \hat{k}_{lmpq}(t)\, W_{lmpq}(\mathbf{R},\ \mathbf{r}^*,\ t\,'),$$

with the Love operator (and, consequently, its kernel, the Love function) bearing dependence upon m, p, and q. Accordingly, (59) would become

$$\bar{U}_{lmpq}(\chi) = \bar{k}_{lmpq}(\chi)\ \bar{W}_{lmpq}(\chi). \tag{70}$$

Fortunately, insofar as the Correspondence Principle is valid, the functional form of $\bar{k}_{lmpq}(\chi)$ depends upon l only and, thus, can be written down simply as $\bar{k}_l(\chi_{lmpq})$. We know this from the considerations offered after equations (144). There we explained that $\bar{k}_l$ depends on $\chi = \chi_{lmpq}$ via $\bar{J}(\chi)$, while *the functional form* of $\bar{k}_l$ bears no dependence on $m,\ p,\ q$.

The phase lag is often denoted as ϵ_{lmpq}, a time-honoured tradition set by Kaula (1964). However, as the lag is expressed through $\bar{k}_l$ via (66), all said above about $\bar{k}_l$ applies to the lag: while the functional form of the dependency $\epsilon_{lmpq}(\chi)$ may be different for different ls, it is invariant under the other three integers, so the notation $\epsilon_l(\chi_{lmpq})$ is more adequate.

It should be mentioned, though, that for bodies of pronounced non-sphericity coupling between the spherical harmonics furnishes the Love numbers and lags whose expressions through the frequency, for a fixed l, have different functional forms for different $m,\ p,\ q$. In these cases, the notations $\bar{k}_{lmpq}$ and ϵ_{lmpq} become necessary (Smith 1974; Wahr 1981a,b,c; Dehant 1987a,b). For a slightly non-spherical body, the Love numbers differ from the Love numbers of the spherical reference body by a term of the order of the flattening, so a small non-sphericity can usually be neglected.

4.5 Rigidity versus self-gravitation

For small bodies and small terrestrial planets, the values of A_l vary from about unity to hundreds. For example, A_2 is about 2 for the Earth, 20 for Mars, 80 for the Moon, and 200 for Iapetus. For superearths, the values will be much smaller than unity, though. Insofar as

$$A_l\ \frac{J}{\mid \bar{J}(\chi) \mid} \gg 1, \tag{71}$$

one can approximate (64) with

$$\bar{k}_l(\chi) = -\frac{3}{2(l-1)}\ \frac{\bar{J}\ (\chi)}{\bar{J}\ (\chi) + A_l\ J} = -\frac{3}{2}\ \frac{\bar{J}\ (\chi)}{A_l\ J} + O\left(\mid \bar{J}/(A_l\ J)\mid^2\right), \tag{72}$$

except in the closest vicinity of an $lmpq$ resonance, where χ_{lmpq} approaches nil and $\bar{J}$ diverges for some rheologies—like, for example, for those of Maxwell or Andrade.

Whenever the approximate formula (72) is applicable, we can rewrite (66) as

$$\tan \epsilon(\chi) \equiv -\frac{\mathcal{I}m\left[\bar{k}_l(\chi)\right]}{\mathcal{R}e\left[\bar{k}_l(\chi)\right]} \approx -\frac{\mathcal{I}m\left[\bar{J}(\chi)\right]}{\mathcal{R}e\left[\bar{J}(\chi)\right]} = \tan \delta(\chi), \tag{73}$$

wherefrom we readily deduce that the phase lag $\epsilon(\chi)$ of the tidal frequency χ coincides with the phase lag of the complex compliance:

$$\epsilon(\chi) \approx \delta(\chi), \tag{74}$$

if χ is not too small (so we are not too close to the commensurability). This way, insofar as the condition (71) is fulfilled, the component $\bar{U}_l(\chi)$ of the primary's potential lags behind the component $\bar{W}_l(\chi)$ of the perturbed potential by the same phase angle as the strain lags behind the stress at frequency χ in a sample of the material. Dependent on the rheology, a vanishing tidal frequency may or may not limit the applicability of (71) and thus cause a considerable difference between ϵ and δ.

In other words, the approximation is valid insofar as changes of shape are determined solely by the local material properties, and not by self-gravitation of the object as a whole. This depends on the rheological model. For a Voigt or SAS[8] solid in the limit of $\chi \to 0$, we have $\bar{J}(\chi) \to J$, so the zero-frequency limit of $\bar{k}_l(\chi)$ is the static Love number $k_l \equiv |\bar{k}(0)|$. In this case, approximation (72–74) remains applicable all the way down to $\chi = 0$. For the Maxwell and Andrade models, however, one obtains, for vanishing frequency: $\bar{J}(\chi) \sim 1/(\eta\chi)$, whence $\bar{\mu} \sim \eta\chi$ and $\bar{k}_2(\chi)$ approaches the hydrodynamical Love number $k_2^{(hyd)} = 3/2$.

We see that for the Voigt and SAS models approximation (74) works, for $A_l \gg 1$, at all frequencies, as the condition $A_l \gg 1$ can be set for any value of χ. For the Maxwell and Andrade solids, the condition holds only at $\chi \gg \tau_{_M}^{-1} A_l^{-1} = \frac{\mu}{\eta} A_l^{-1}$, and so does the approximation (74). At frequencies below this threshold, self-gravitation "beats" the local material properties, and the behaviour of the tidal lag deviates from that of the lag in a sample. This fact should be kept in mind when one wants to explore crossing of a resonance.

A standard caveat is in order, concerning formulae (72–74). As in reality the potential $\bar{U}$ is a function of ω and not χ, our illegitimate use of χ should be compensated by multiplying $\epsilon_l(\chi_{lmpq})$ with sgn ω_{lmpq}, when ϵ_l shows up in the expression for the tidal force or torque.

4.6 The case of inhomogeneous bodies

Tidal dissipation within a multilayer near-spherical body is studied through expanding the involved fields over the spherical harmonics in each layer, setting the boundary conditions on the outer surface, and using the matching conditions on boundaries between layers. This formalism was developed by Alterman et al. (1959). An updated discussion of the method can be found in Sabadini and Vermeersen (2004). For a brief review, see Legros et al. (2006).

Calculation of tidal dissipation in a Jovian planet is an even more formidable task (see Remus et al. 2012a and references therein). However dissipation in a giant planet with a solid core may turn out to be approachable by analytic means (Remus et al. 2011, 2012b).

[8] The acronym *SAS* stands for the *Standard Anelastic Solid*, also known as the Hohenemser-Prager model.

5 Dissipation at different frequencies

5.1 The data collected on the Earth: in the lab, over seismological basins, and through geodetic measurements

In Efroimsky and Lainey (2007), we considered the generic rheological model

$$Q = (\mathcal{E}\,\chi)^{\alpha}\,, \tag{75a}$$

where χ is the tidal frequency and $\mathcal{E}$ is a parameter having the dimensions of time. The physical meaning of this parameter is elucidated in Efroimsky and Lainey (2007). Under the special choice of $\alpha = -1$ and for sufficiently large values of Q, this parameter coincides with the time lag Δt which, for this special rheology, turns out to be the same at all frequencies.

Actual experiments register not the inverse quality factor but the phase lag between the reaction and the action. So the empirical law should rather be written down as

$$\frac{1}{\sin\delta} = (\mathcal{E}\,\chi)^{\alpha}\,, \tag{75b}$$

which is equivalent to (75a), provided the Q factor is defined there as Q_{energy} and not as Q_{work}—see Sect. 2.2 for details.

The applicability realm of the empirical power law (75) is remarkably broad—in terms of both the physical constituency of the bodies and their chemical composition. Most intriguing is the robust universality of the values taken by the index α for very different materials: between 0.2 and 0.4 for ices and silicates, and between 0.14 and 0.2 for partial melts. Historically, two communities independently converged on this form of dependence.

In the material sciences, the rheological model (82), wherefrom the power law (75b) stems, traces its lineage to the groundbreaking work by Andrade (1910) who explored creep in metals. Through the subsequent century, this law was found to be applicable to a vast variety of other materials, including minerals (Weertman and Weertman 1975; Tan et al. 1997) and their partial melts (Fontaine et al. 2005). With almost the same values of α, this law applies also to ices (McCarthy et al. 2007; Castillo-Rogez 2009), a milestone result, taken the physical and chemical differences between ices and silicates. It is agreed upon that in crystalline materials the Andrade regime can find its microscopic origin both in the dynamics of dislocations (Karato and Spetzler 1990) and in the grain-boundary diffusional creep (Gribb and Cooper 1998). As the same behaviour is inherent in metals, silicates, ices, and even glass-polyester composites (Nechada et al. 2005), it should stem from a single underlying phenomenon determined by some principles more general than specific material properties. An attempt to find such a universal mechanism was undertaken by Miguel et al. (2002). See also the theoretical considerations offered in Karato and Spetzler (1990).

In seismology, the power law (75) became popular in the second part of the XXth century, with the progress of precise measurements on large seismological basins (Mitchell 1995; Stachnik et al. 2004; Shito et al. 2004). Further confirmation of this law came from geodetic experiments that included: (a) satellite laser ranging (SLR) measurements of tidal variations in the J_2 component of the gravity field of the Earth; (b) space-based observations of tidal variations in the Earth's rotation rate; and (c) space-based measurements of the Chandler wobble period and damping (Benjamin et al. 2006; Eanes and Bettadpur 1996; Eanes 1995). Not surprisingly, the Andrade law became a key element in the recent attempt to construct a universal rheological model of the Earth's mantle (Birger 2007). This law also became a component of the non-hydrostatic-equilibrium model for the zonal tides in an inelastic Earth

by Defraigne and Smits (1999), a model that became the basis for the IERS Conventions (Petit and Luzum 2010). While the lab experiments give for α values within 0.2–0.4, the geodetic techniques favour the interval 0.14–0.2. This minor discrepancy may have emerged due to the presence of partial melt in the mantle and, possibly, due to nonlinearity at high bounding pressures in the lower mantle. The universality of the Andrade law compels us to assume that (75) is generic for silicate planets and moons. Similarly, the applicability of (75) to samples of ices in the lab is likely to indicate that this law can be employed for description of an icy moon as a whole.

Karato and Spetzler (1990) argue that at frequencies below a certain threshold χ_0 anelasticity gives way to purely viscoelastic behaviour, so the parameter α becomes close to unity. This detail is missing in the theory of the Chandler wobble of Mars, by Zharkov and Gudkova (2009).[9] For the Earth's mantle, the threshold corresponds to the time-scale about a year or slightly longer. Although in Karato and Spetzler (1990) the rheological law is written in terms of $1/Q$, we shall substitute it with a law more appropriate to the studies of tides:

$$k_l \;\sin\epsilon_l = (\,\mathcal{E}\,\chi\,)^{-p}\,, \qquad \text{where} \quad p = 0.2 - 0.4 \text{ for } \chi \;>\; \chi_0 \quad \text{and} \quad p \sim 1 \text{ for } \chi < \chi_0, \tag{76}$$

χ being the frequency, and χ_0 being the frequency threshold below which viscosity takes over anelasticity.

The reason why we write the power scaling law as (76) and not as (75) is that at the lowest frequencies the geodetic measurements render $k_l\;\sin\epsilon_l$, not the lag angle δ in a sample (e.g., Benjamin et al. 2006). For the same reason, we denoted the exponents in (75) and (76) with different letters, α and p. Below we shall see that these exponents do not always coincide. Another reason for giving preference to (76) is that not only the sine of the lag but also the absolute value of the Love number is frequency-dependent.

5.2 Tidal damping in the Moon, from laser ranging

Fitting of the LLR data to the scaling law (75) was carried out by Williams et al. (2001) who demonstrated that the lunar mantle possesses quite an abnormal value of the exponent: -0.19. A later reexamination in Williams et al. (2008) gave a less embarrassing value, -0.09, which was still negative and thus seemed to contradict our knowledge about microphysical damping mechanisms in minerals. Thereupon, Williams and Boggs (2009) commented:

> "There is a weak dependence of tidal specific dissipation Q on period. The Q increases from $\sim$30 at a month to $\sim$35 at one year. Q for rock is expected to have a weak dependence on tidal period, but it is expected to decrease with period rather than increase. The frequency dependence of Q deserves further attention and should be improved."

While there always remains a possibility of the raw data being insufficient or of the fitting procedure being imperfect, the fact is that the negative exponent obtained in Williams and Boggs (2009) does **not** necessarily contradict the law (75). Indeed, the exponent obtained through LLR was not the α from (75) but was the p from (76). The distinction is critical due to the difference in frequency-dependence of the seismic and tidal dissipation. It turns out that the near-viscous value $p \sim 1$ from the second line of (76), appropriate for low frequencies, does not retain its value all the way to the zero frequency. Specifically, in Sect. 5.4 we

[9] This circumstance was ignored also by Defraigne and Smits (1999). Accordingly, if the claims by Karato and Spetzler (1990) are correct, the table of corrections for the tidal variations in the Earth's rotation in the IERS Conventions is likely to contain increasing errors for periods of about a year and longer.

shall see that below the frequency $1/(\tau_M\, A_l)$ the exponential p begins to decrease with the decrease of the frequency. (Here $\tau_M = \eta/\mu$ is the Maxwell time, η and μ being the mantle's viscosity and rigidity.) As the frequency becomes lower, p changes its sign and eventually becomes -1 in a close vicinity of $\chi = 0$. This behaviour follows from calculations based on a realistic rheology (formulae 88–90 below), and goes along with the evident physical fact that the average tidal torque must vanish in a resonance.[10] In Sect. 5.7, comparison of this behaviour with the LLR results will yield us an estimate for the mean lunar viscosity.

5.3 The Andrade model as an example of viscoelastic behaviour

The complex compliance of a Maxwell material contains a term $J = J(0)$ responsible for the elastic part of the deformation and a term $-i/\chi\eta$ describing the viscosity. Any additional term in the compliance will correspond to other forms of hereditary reaction. The available geophysical data strongly favour a particular extension of the Maxwell approach, the *Andrade model* (Cottrell and Aytekin 1947; Duval 1976). In modern notations, the model looks as [11]

$$J(t-t\,') = \left[J + \beta\,(t-t\,')^{\alpha} + \eta^{-1}\left(t-t\,'\right)\right]\,\Theta(t-t\,'), \tag{77}$$

where α is a dimensionless parameter, β is a dimensional parameter, η is the steady-state viscosity, while J is the unrelaxed compliance which is inverse to the unrelaxed rigidity: $J \equiv J(0) = 1/\mu(0) = 1/\mu$. So (77) is the Maxwell model amended with a hereditary term.

An example illustrating the model is rendered by deformation under constant loading. The anelastic term dominates at short times, the strain being a convex function of t (the so-called primary or transient creep). As time goes on and the load is kept constant, the viscous term becomes larger, and the strain becomes almost linear in t (the secondary creep).

For all minerals (including ices), α assumes values from 0.14 to 0.4 (more often, through 0.3)—see the references in Sect. 5.1. The parameter β may be rewritten as

$$\beta = J\,\tau_{_A}^{-\alpha} = \mu^{-1}\,\tau_{_A}^{-\alpha}, \tag{78}$$

the quantity $\tau_{_A}$ having dimensions of time. This timescale associated with the Andrade creep may be termed as the "Andrade time" or the "anelastic time". We see from (78) that a short $\tau_{_A}$ makes the anelasticity more pronounced, while a long $\tau_{_A}$ makes the anelasticity weak.[12]

Within some frequency bands, the Andrade time gets very close to the Maxwell time (Castillo-Rogez et al. 2011; Castillo-Rogez and Choukroun 2010):

$$\tau_{_A} \approx \tau_{_M} \Longrightarrow \beta \approx J\,\tau_{_M}^{-\alpha} = J^{1-\alpha}\,\eta^{-\alpha} = \mu^{\alpha-1}\,\eta^{-\alpha}, \tag{79}$$

where the viscoelastic timescale (the relaxation Maxwell time) is given by

$$\tau_{_M} \equiv \frac{\eta}{\mu} = \eta\,J. \tag{80}$$

[10] For example, the principal tidal torque $\tau_{lmpq} = \tau_{2200}$ acting on a secondary must vanish on crossing the synchronous orbit. This indeed happens, as the exponential p becomes -1 in the close vicinity of $\chi_{2200} = 0$.

[11] As we integrate over $t-t\,' \in [\,0,\ \infty\,)$, the terms $\beta(t-t\,')^{\alpha}$ and $\eta^{-1}\left(t-t\,'\right)$ can do without the Heaviside step-function $\Theta(t-t\,')$. We remind though that the first term, J, does need this multiplier, so that insertion of (77) into (39) renders the desired $J\,\delta(t-t\,')$ under the integral, after the differentiation in (39) is performed.

[12] While the Andrade creep is caused by "unpinning" of jammed dislocations (Karato and Spetzler 1990; Miguel et al. 2002), it is not apparently clear if $\tau_{_A}$ can be identified with the typical time of unpinning.

One cannot expect $\tau_{_A}$ and $\tau_{_M}$ to coincide in all situations, as they may possess some degree of frequency-dependence. There exist indications that in the Earth's mantle the role of anelasticity (compared to viscoelasticity) undergoes a decrease when the frequencies become lower than 1/year—see the miscrophysical model suggested in section 5.2.3 of Karato and Spetzler (1990). The relation between $\tau_{_A}$ and $\tau_{_M}$ may depend also upon the intensity of loading, i.e., upon the damping mechanisms involved. The microphysical model considered in Karato and Spetzler (1990) was applicable to strong deformations, with anelastic dissipation being dominated by dislocations unpinning. Accordingly, the dominance of viscosity over anelasticity ($\tau_{_A} \ll \tau_{_M}$) at low frequencies may be regarded proven for strong deformations only. At low stresses, when the grain-boundary diffusion mechanism is dominant, the values of $\tau_{_A}$ and $\tau_{_M}$ may remain comparable at low frequencies. The topic needs further research.

In terms of the Andrade and Maxwell times, the compliance becomes:

$$J(t-t\,') = J\left[1+\left(\frac{t-t\,'}{\tau_{_A}}\right)^{\alpha}+\frac{t-t\,'}{\tau_{_M}}\right]\Theta(t-t\,'). \tag{81}$$

In the frequency domain, compliance (81) will look:

$$\bar{J}(\chi) = J+\beta\,(i\chi)^{-\alpha}\;\Gamma\,(1+\alpha)-\frac{i}{\eta\chi} \tag{82a}$$

$$= J\,\left[1+(i\;\chi\;\tau_{_A})^{-\alpha}\;\Gamma\,(1+\alpha)-i\;(\chi\;\tau_{_M})^{-1}\right], \tag{82b}$$

where χ is the frequency and Γ is the Gamma function. The imaginary and real parts are:

$$\mathcal{I}m[\bar{J}(\chi)] = -(\eta\,\chi)^{-1}-\chi^{-\alpha}\,\beta\;\sin\,(\alpha\,\pi/2)\;\;\Gamma(\alpha+1) \tag{83a}$$

$$= -J\,(\chi\tau_{_M})^{-1}-J\,(\chi\tau_{_A})^{-\alpha}\;\;\sin\,(\,\alpha\,\pi/2\,)\;\;\Gamma(\alpha+1), \tag{83b}$$

$$\mathcal{R}e[\bar{J}(\chi)] = J+\chi^{-\alpha}\,\beta\;\cos\,(\alpha\,\pi/2)\;\;\Gamma(\alpha+1) \tag{84a}$$

$$= J+J\,(\chi\tau_{_A})^{-\alpha}\;\;\cos\,(\alpha\,\pi/2)\;\;\Gamma(\alpha+1), \tag{84b}$$

whence we obtain the following dependence of the phase lag upon the frequency:

$$\tan\delta(\chi) = -\frac{\mathcal{I}m\left[\bar{J}(\chi)\right]}{\mathcal{R}e\left[\bar{J}(\chi)\right]} = \frac{(\eta\;\chi)^{-1}+\chi^{-\alpha}\;\;\beta\;\;\sin\,(\alpha\,\pi/2)\,\Gamma\,(\alpha+1)}{\mu^{-1}+\chi^{-\alpha}\;\;\beta\;\;\cos\,(\alpha\,\pi/2)\;\;\Gamma\,(\alpha+1)} \tag{85a}$$

$$= \frac{z^{-1}\,\zeta+z^{-\alpha}\;\;\sin\,(\alpha\,\pi/2)\,\Gamma\,(\alpha+1)}{1+z^{-\alpha}\;\;\cos\,(\alpha\;\pi/2)\;\;\Gamma\,(\alpha+1)}. \tag{85b}$$

Here z is the dimensionless frequency defined as

$$z \equiv \chi\;\tau_{_A} = \chi\;\tau_{_M}\;\zeta, \tag{86}$$

while ζ is a dimensionless parameter of the Andrade model:

$$\zeta \equiv \tau_{_A}/\tau_{_M}. \tag{87}$$

5.4 Tidal response of viscoelastic spherical bodies obeying the Andrade and Maxwell models

An $lmpq$ term of the tidal torque is proportional to $k_l(\chi)\;\sin\epsilon_l(\chi) \;=\; |\bar{k}_l(\chi_{lmpq})|\;\sin\;\epsilon_l(\chi_{lmpq})$. These factors' frequency dependence can be obtained by combining (68) with (82). Leaving details for Appendix C.2, we present the results, without the sign multiplier.

- *The high-frequency band:* $\chi \gg \chi_{HI}$,

$$|\bar{k}_l(\chi)|\,\sin\epsilon_l(\chi) \approx \frac{3}{2\,(l-1)}\,\frac{A_l}{(A_l+1)^2}\,\sin\left(\frac{\alpha\pi}{2}\right)\,\Gamma(\alpha+1)\,\zeta^{-\alpha}\,\left(\tau_{_M}\,\chi\right)^{-\alpha}. \quad (88)$$

At high frequencies, anelasticity dominates. So, dependent upon the microphysics of the mantle, the parameter ζ may be of order unity or slightly lower. We say *slightly*, because we expect both anelasticity and viscosity to be present near the transitional zone. (A too low ζ would eliminate viscosity from the picture.) Were the parameter ζ evolving slowly with the decrease of the frequency, we would be able to say that for small bodies and small terrestrial planets (i.e., for $A_l \gg 1$), the boundary between the high and intermediate frequencies is

$$\chi_{HI} = \tau_{_M}^{-1}\,\zeta^{\frac{\alpha}{1-\alpha}},$$

while for large terrestrial planets (i.e., for $A_l \ll 1$) the boundary is

$$\chi_{HI} = \tau_{_A}^{-1} = \tau_{_M}^{-1}\,\zeta^{-1}.$$

Both these formulae are derived in the Appendix.

In reality, these two estimates may be overruled by microphysics. At a certain threshold frequency, the anelastic timescale $\tau_{_A}$ and therefore the parameter ζ begin to grow rapidly with the decrease of the frequency. This happens when the effectiveness of anelastic dissipation (defect unpinning) begins to decrease. Whenever this threshold happens to be higher than the above estimates for χ_{HI}, one has simply to identify χ_{HI} with the said threshold. According to Karato and Spetzler (1990), for the Earth's mantle this threshold is about 1/year. Hence for terrestrial planets it may be more realistic to set, by hand:

$$\chi_{HI} \approx 1\,/\text{year}.$$

- *The intermediate-frequency band:* $\chi_{HI} \gg \chi \gg \tau_{_M}^{-1}\,(A_l+1)^{-1}$,

$$|\bar{k}_l(\chi)|\,\sin\epsilon_l(\chi) \approx \frac{3}{2\,(l-1)}\,\frac{A_l}{(A_l+1)^2}\,\left(\tau_{_M}\,\chi\right)^{-1}. \quad (89)$$

While Appendix C.2 renders $\tau_{_M}^{-1}\,\zeta^{\frac{\alpha}{1-\alpha}}$ for the upper bound, here we identify this bound with the afore-mentioned threshold χ_{HI}, in understanding that below this frequency ζ begins to grow rapidly, thus causing a switch from the Andrade to a near-Maxwell regime.

- *The low-frequency band:* $\tau_{_M}^{-1}\,(A_l+1)^{-1} \gg \chi$,

$$|\bar{k}_l(\chi)|\,\sin\epsilon_l(\chi) \approx \frac{3}{2\,(l-1)}\,A_l\,\tau_{_M}\,\chi. \quad (90)$$

Scaling laws (88) and (89) mimic, up to constant factors, the frequency-dependencies of $|\bar{J}(\chi)|\,\sin\delta(\chi) = -\mathcal{I}m[\bar{J}(\chi)]$ at high and low frequencies, correspondingly, see Eq. (83).

Expression (90) however shows a remarkable phenomenon inherent only in the *tidal* lagging, and not in the lagging in a sample of material. The $|\bar{k}_l(\chi)|\,\sin\epsilon_l(\chi)$ factor changes its behaviour drastically on close approach to the zero frequency. Having reached a finite maximum at about $\chi = \tau_{_M}^{-1}(A_l+1)^{-1}$, the said factor begins to scale linearly in χ as χ approaches zero. This way, $|\bar{k}_l(\chi)|\,\sin\epsilon_l(\chi)$ decreases continuously on close approach to a resonance and goes through nil together with the frequency. So neither the tidal torque nor the tidal force explodes in resonances. In a somewhat heuristic manner, this change in the frequency-dependence was pointed out, for $l = 2$, in Efroimsky and Williams (2009).

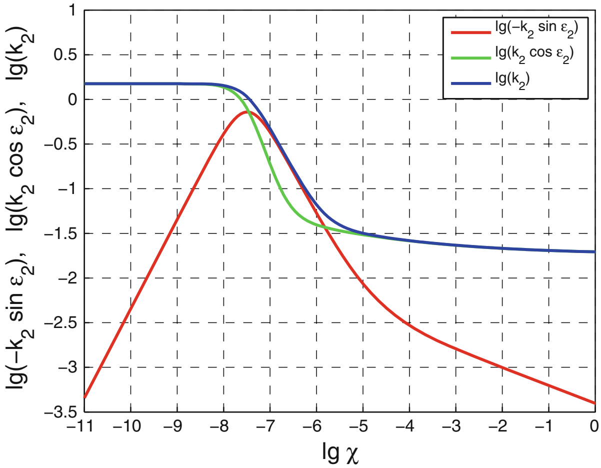

Fig. 1 Tidal response of a homogeneous spherical Andrade body, set against the decadic logarithm of the forcing frequency χ (in Hz). The *blue curve* renders the decadic logarithm of the absolute value of the quadrupole complex Love number, $\lg k_2 = \lg|\bar{k}_2(\chi)|$. The *green* and *red curves* depict the logarithms of the real and the negative imaginary parts of the Love number: $\lg \mathcal{R}e[\bar{k}_2(\chi)] = \lg(k_2 \cos\epsilon_2)$ and $\lg\{-\mathcal{I}m[\bar{k}_2(\chi)]\} = \lg(-k_2 \sin\epsilon_2)$, accordingly. The change in the slope of the *red curve* (the "elbow"), which takes place to the right of the maximum, corresponds to the switch from viscosity dominance at lower frequencies to anelasticity dominance at higher frequencies. The parameters A_2 and $\tau_{_M}$ were given values appropriate to a homogeneous Moon with a low viscosity, as described in Sect. 5.7. The plots were generated for an Andrade body with $\zeta = 1$ at all frequencies. Setting the body Maxwell at lower frequencies will only slightly change the shape of the "elbow" and will have virtually no effect on the maximum

In that paper, we also tried to describe, through semi-qualitative reasoning, the frequency-dependence of the Love number k_2. We concluded that it should be proportional to the cosine of the phase lag. Accurate treatment shows that for the Kelvin–Voigt model this is true over all frequencies. For the Maxwell model, this relation works in the low-frequency limit. For details, see the Electronic Supplementary Material 5 "The behaviour of $k_l(\chi) \equiv |\bar{k}_l(\chi)|$ in the limit of vanishing tidal frequency χ, within the Andrade and Maxwell models".

5.5 Example

Figure 1 shows the absolute value, $k_2 \equiv |\bar{k}_2(\chi)|$, the real part, $k_2 \cos\epsilon_2 = \mathcal{R}e\left[\bar{k}_2(\chi)\right]$, and the negative imaginary part, $k_2 \sin\epsilon_2 = -\mathcal{I}m\left[\bar{k}_2(\chi)\right]$, of the quadrupole Love number. Each quantity's decadic logarithm is plotted against the decadic logarithm of the frequency χ (in Hz). The curves were obtained by insertion of formulae (83–84) into (64). As an example, the case of $-\mathcal{I}m\left[\bar{k}_2(\chi)\right]$ is worked out in Appendix C.2, formulae (168–170).

Both in the high- and low-frequency limits, the negative imaginary part of $\bar{k}_2(\chi)$ (the red curve in Fig. 1) approaches zero. Accordingly, in the low- and high-frequency bands the real part (the green curve) virtually coincides with the absolute value (the blue curve).

While on the left and on the close right of the peak, dissipation is mainly due to viscosity, friction at higher frequencies is mainly due to anelasticity. This switch corresponds to the change of the slope of the red curve at high frequencies (for our choice of parameters, at around 10^{-5} Hz). This change of the slope is often called *the elbow*.

Figure 1 was generated for $A_2 = 80.5$ and $\tau_{_M} = 3.75 \times 10^5$ s. The value of A_2 corresponds to the Moon modeled by a homogeneous sphere of rigidity $\mu = 0.8 \times 10^{11}$ Pa. Our choice of the value of $\tau_{_M} \equiv \eta/\mu$ corresponds to a homogeneous Moon with the said value of rigidity

and with viscosity set to be $\eta = 3.75 \times 10^5$ Pa s. The reason why we consider an example with such a low value of η will be explained in Sect. 5.7. Finally, it was assumed for simplicity that $\zeta = 1$, i.e., that $\tau_{_A} = \tau_{_M}$. Although unphysical at low frequencies, this simplification only slightly changes the shape of the "elbow" and exerts virtually no influence upon the maximum of the red curve, provided the maximum is located well into the viscosity zone.

5.6 Crossing a resonance—with a chance for entrapment

In the expansion for the tidal torque, factors (88–90) appear in the company of sgn ω. For example, the factor (90) enters the $lmpq$ term of the torque as

$$|\bar{k}_l(\chi_{lmpq})| \sin\epsilon_l(\chi_{lmpq}) \;\mathrm{sgn}\,\omega_{lmpq}$$
$$\approx \frac{3}{2\,(l-1)}\, A_l\, \tau_{_M}\, \chi_{lmpq}\;\mathrm{sgn}\,\omega_{lmpq} = \frac{3}{2\,(l-1)}\, A_l\, \tau_{_M}\, \omega_{lmpq}\,. \tag{91}$$

Naturally, this term depends on ω_{lmpq}, not on $\chi_{lmpq} = |\omega_{lmpq}|$. Together with ω_{lmpq}, the term goes continuously through zero, and changes its sign as the $lmpq$ resonance is crossed. This gets along with the physically evident fact that the principal (i.e., 2200) term of the tidal torque should vanish as the secondary approaches the synchronous orbit.

An $lmpq$ term of the torque changes its sign and thus creates a chance for entrapment. As the value of an $lmpq$ term of the torque is much lower than that of the principal, 2200 term, a perfectly spherical body will never get stuck in a resonance other than 2200. (The latter is, of course, the 1 : 1 resonance in which the principal term of the torque vanishes.) However, the presence of the triaxiality-generated torque is known to contribute to the probabilities of entrapment into other resonances (provided the eccentricity is not zero). Typically, in the literature they consider a superposition of the triaxiality-generated torque with the principal tidal term. We would point out that the "trap" shape of the $lmpq$ term (91) makes this term relevant for the study of entrapment in the $lmpq$ resonance. In some situations, one has to take into account also the non-principal terms of the tidal torque.

5.7 Comparison with the LLR results

As we mentioned above, fitting of the LLR data to the power law has resulted in a very small *negative* exponential $p = -0.19$ (Williams et al. 2001). As the measurements of the lunar damping described in Williams et al. (2001) rendered information on the *tidal* and not seismic dissipation, those results can and should be compared to the scaling law (88–90). As the small negative exponential was devised from observations over periods of a month to a year, it is natural to presume that the appropriate frequencies were close to or slightly below the frequency $\dfrac{1}{\tau_{_M}\,(A_2+1)}$ at which the factor $k_2\,\sin\epsilon_2$ has its peak, as in Fig. 1:

$$3\times 10^6\,\mathrm{s} \approx 0.1\,\mathrm{year} = \tau_{_M}\,(A_2+1) \approx \tau_{_M}\,A_2 = \frac{\eta}{\mu}\,A_2 = \frac{57\,\eta}{8\,\pi\,G\,(\rho\,R)^2}\,. \tag{92}$$

Hence, were the Moon a uniform viscoelastic body, its viscosity would be only

$$\eta = 3\,\times\,10^{16}\,\mathrm{Pa\;\;s}. \tag{93}$$

Hence the lower lunar mantle contains a high percentage of partial melt, a fact which goes along with the model by Weber et al. (2011), and which was anticipated in Williams et al. (2001) and Williams and Boggs (2009) following an earlier work by Nakamura et al. (1974).

6 The polar component of the tidal torque acting on the primary

Let vector $\mathbf{r} = (r,\ \lambda,\ \phi)$ point from the centre of the primary toward a point-like secondary of mass M_{sec}. Associating the coordinate system with the primary, we reckon the latitude ϕ from the equator. Setting the coordinate system to corotate, we determine the longitude λ from a fixed meridian. The tidally induced component of the primary's potential, U, can be generated either by this secondary itself or by some other secondary of mass M^*_{sec} located at $\mathbf{r}^* = (r^*,\ \lambda^*,\ \phi^*)$. In either situation, the tidally induced potential U generates a tidal force and a tidal torque wherewith the secondary of mass M_{sec} acts on the primary.

The scope of this paper is limited to low i. When the role of the primary is played by a planet, the secondary being its satellite, i is the satellite's inclination. When the role of the primary is played by the satellite, the planet acting as its secondary, i denotes the satellite's obliquity. Similarly, when the planet is regarded as a primary and its host star is treated as its secondary, i is the obliquity of the planet. In all these cases, the smallness of i indicates that the tidal torque acting on the primary is reduced to its polar component. The other components of the torque will be neglected in this approximation.

The polar component of the torque acting on the primary is the negative of the partial derivative of the tidal potential, with respect to the primary's sidereal angle:

$$\mathcal{T}(\mathbf{r}) = -M_{sec}\ \frac{\partial U(\mathbf{r})}{\partial \theta}, \tag{94}$$

θ standing for the primary's sidereal angle. The formula is convenient when the tidal potential U is expressed through the secondary's orbital elements and the primary's sidereal angle.[13]

Here and hereafter we deliberately refer to *a primary and a secondary* in lieu of *a planet and a satellite*. The preference stems from our intention to extend the formalism to setting where a moon is playing the role of a tidally-perturbed primary, the planet being its tide-producing secondary. Similarly, when addressing the rotation of Mercury, we interpret the Sun as a secondary that is causing a tide on the effectively primary body, Mercury.

7 The tidal potential

7.1 Darwin (1879) and Kaula (1964)

The potential produced at point $\mathbf{R} = (R, \lambda, \phi)$ by a secondary body of mass M^*, located at $\mathbf{r}^* = (r^*,\ \lambda^*, \phi^*)$ with $r^* \geq R$, is given by (1). When a tide-raising secondary located at $\mathbf{r}^*$ distorts the shape of the primary, the potential generated by the primary at some exterior point $\mathbf{r}$ acquires an amendment given by (2). Insertion of (1) into (2) entails

[13] Were the potential written down in the spherical coordinates associated with the primary's equator and corotating with the primary, the polar component of the tidal torque could be calculated with aid of the expression

$$\mathcal{T}(\mathbf{r}) =\ M_{sec}\ \frac{\partial U(\mathbf{r})}{\partial \lambda}$$

derived, for example, in Williams and Efroimsky (2012). That the expression agrees with (94) can be seen from the formula

$$\lambda = -\theta + \Omega + \omega + \nu + O(i^2) = -\theta + \Omega + \omega + \mathcal{M} + 2\ e\ \sin \mathcal{M} + O(e^2) + O(i^2),$$

$e, i, \omega, \Omega, \nu$ and $\mathcal{M}$ being the eccentricity, inclination, argument of the pericentre, longitude of the node, true anomaly, and mean anomaly of the tide-raising secondary.

$$U(\mathbf{r}) = -\,GM^*_{sec}\sum_{l=2}^{\infty}\frac{k_l\;R^{2l+1}}{r^{l+1}\;{r^*}^{l+1}}\sum_{m=0}^{l}\frac{(l-m)!}{(l+m)!}$$
$$\times\,(2-\delta_{0m})\,P_{lm}(\sin\phi)\,P_{lm}(\sin\phi^*)\;\cos m(\lambda-\lambda^*). \tag{95}$$

A more practical formula was offered by Kaula (1961, 1964), who switched from the spherical coordinates to the Kepler elements $(\,a^*,\;e^*,\;i^*,\;\Omega^*,\;\omega^*,\;\mathcal{M}^*\,)$ and $(\,a,\;e,\;i,\;\Omega,\;\omega,\;\mathcal{M}\,)$ of the secondaries located at $\mathbf{r}^*$ and $\mathbf{r}$. Application of this technique to (95) results in

$$U(\mathbf{r}) = -\sum_{l=2}^{\infty}k_l\,\left(\frac{R}{a}\right)^{l+1}\,\frac{GM^*_{sec}}{a^*}\,\left(\frac{R}{a^*}\right)^{l}\sum_{m=0}^{l}\frac{(l-m)!}{(l+m)!}\,(2-\delta_{0m})\sum_{p=0}^{l}F_{lmp}(i^*)$$
$$\sum_{q=-\infty}^{\infty}G_{lpq}(e^*)\sum_{h=0}^{l}F_{lmh}(i)\sum_{j=-\infty}^{\infty}G_{lhj}(e)\;\cos\left[\left(v^*_{lmpq}-m\theta^*\right)-\left(v_{lmhj}-m\theta\right)\right]\;, \tag{96}$$

where

$$v^*_{lmpq}\equiv(l-2p)\omega^*+(l-2p+q)\mathcal{M}^*+m\,\Omega^*, \tag{97}$$
$$v_{lmhj}\equiv(l-2h)\omega+(l-2h+j)\mathcal{M}+m\,\Omega, \tag{98}$$

$\theta=\theta^*$ being the sidereal angle, $G_{lpq}(e)$ signifying the eccentricity functions,[14] and $F_{lmp}(i)$ denoting the inclination functions (Gooding and Wagner 2008).

Being equivalent for a planet with an instant response of the shape, (95) and (96) disagree when friction-caused delays come into play. Kaula's expression (96), as well as its truncated, Darwin's, version,[15] is capable of accommodating separate phase lags for each mode:

$$U(\mathbf{r}) = -\sum_{l=2}^{\infty}k_l\,\left(\frac{R}{a}\right)^{l+1}\,\frac{GM^*_{sec}}{a^*}\,\left(\frac{R}{a^*}\right)^{l}\sum_{m=0}^{l}\frac{(l-m)!}{(l+m)!}\,(2-\delta_{0m})\sum_{p=0}^{l}F_{lmp}(i^*)$$
$$\sum_{q=-\infty}^{\infty}G_{lpq}(e^*)\sum_{h=0}^{l}F_{lmh}(i)\sum_{j=-\infty}^{\infty}G_{lhj}(e)\;\cos\left[\left(v^*_{lmpq}-m\theta^*\right)-\left(v_{lmhj}-m\theta\right)-\epsilon_{lmpq}\right], \tag{99}$$

where

$$\epsilon_{lmpq}=\left[(l-2p)\,\dot{\omega}^*+(l-2p+q)\,\dot{\mathcal{M}}^*+m\,(\dot{\Omega}^*-\dot{\theta}^*)\,\right]\,\Delta t_{lmpq}$$
$$=\omega^*_{lmpq}\,\Delta t_{lmpq}=\pm\,\chi^*_{lmpq}\,\Delta t_{lmpq} \tag{100}$$

is the phase lag. The tidal mode ω^*_{lmpq} introduced in (100) is

$$\omega^*_{lmpq}\equiv(l-2p)\;\dot{\omega}^*+(l-2p+q)\;\dot{\mathcal{M}}^*+m\;(\dot{\Omega}^*-\dot{\theta}^*), \tag{101}$$

while the positively-defined quantity

---

[14] Functions $G_{lhj}(e)$ coincide with the Hansen polynomials $X^{(-l-1),\,(l-2p)}_{(l-2p+q)}(e)$. In the Electronic Supplementary Material 2 "*The eccentricity functions*", we present a table of the $G_{lhj}(e)$ functions, which are needed for expanding the tides up to terms with e^6, inclusively.

[15] While the treatment by Kaula (1964) entails the infinite Fourier series (96), the development by Darwin (1879) renders its partial sum with $|l|,\;|q|,\;|j|\;\leq\;2$. For introduction into Darwin's method see Ferraz-Mello et al. (2008). Be mindful that these authors' convention on the notations $\mathbf{r}$ and $\mathbf{r}^*$ is opposite to ours.

$$\chi^*_{lmpq} \equiv |\,\omega^*_{lmpq}\,| = |\,(l-2p)\,\dot{\omega}^* + (l-2p+q)\,\dot{\mathcal{M}}^* + m\,(\dot{\Omega}^* - \dot{\theta}^*)\,| \tag{102}$$

is the actual physical $lmpq$ frequency excited by the tide in the primary. The corresponding positively-defined time delay $\Delta t_{lmpq} = \Delta t_l(\chi_{lmpq})$ depends on this physical frequency, the functional forms of this dependence being different for different materials.

In neglect of the apsidal and nodal precession and also of $\dot{\mathcal{M}}_0$, the above formulae look:

$$\omega_{lmpq} = (l-2p+q)\,n - m\,\dot{\theta}, \tag{103}$$

$$\chi_{lmpq} \equiv |\,\omega_{lmpq}\,| = |\,(l-2p+q)\,n - m\,\dot{\theta}\,|, \tag{104}$$

$$\epsilon_{lmpq} \equiv \omega_{lmpq}\,\Delta t_{lmpq} = \left[(l-2p+q)\,n - m\,\dot{\theta}\,\right]\,\Delta t_{lmpq} \tag{105a}$$

$$= \chi_{lmpq}\,\Delta t_l(\chi_{lmpq})\,\mathrm{sgn}\,\left[(l-2p+q)\,n - m\,\dot{\theta}\,\right], \tag{105b}$$

Formulae (96) and (99) constitute the pivotal result of Kaula's theory of tides. Importantly, this theory imposes no a priori constraint on the form of frequency-dependence of the lags.

8 The Darwin torque

The popular model of entrapment into the 1:1 resonance, developed by Goldreich (1966), rests on the MacDonald torque and thus implies an unphysical rheology of the satellite's material (Williams and Efroimsky 2012). The rheology is given by (75) with $\alpha = -1$. More realistic is the dissipation law (76). An even more accurate and practical formulation of the damping law, stemming from the Andrade formula for the compliance, is rendered by (88–90). These formulae should be inserted into the Darwin-Kaula theory of tides.

8.1 The secular and oscillating parts of the Darwin torque

8.1.1 The general formula

Direct differentiation of (99) with respect to $-\theta$ will result in the expression[16]

$$\mathcal{T} = -\sum_{l=2}^{\infty}\left(\frac{R}{a}\right)^{l+1}\frac{G M^*_{sec}\,M_{sec}}{a^*}\left(\frac{R}{a^*}\right)^{l}\sum_{m=0}^{l}\frac{(l-m)!}{(l+m)!}\,2\,m\sum_{p=0}^{l}F_{lmp}(i^*)$$
$$\sum_{q=-\infty}^{\infty}G_{lpq}(e^*)\sum_{h=0}^{l}F_{lmh}(i)\sum_{j=-\infty}^{\infty}G_{lhj}(e)k_l\,\sin\left[v^*_{lmpq} - v_{lmhj} - \epsilon_{lmpq}\right]. \tag{106}$$

If the tidally-perturbed and tide-raising secondaries are the same body, then $M_{sec} = M^*_{sec}$, and all the elements coincide with their counterparts with an asterisk. Hence the differences

$$v^*_{lmpq} - v_{lmhj}$$
$$= (l-2p+q)\,\mathcal{M}^* - (l-2h+j)\,\mathcal{M} + m\,(\Omega^* - \Omega) + l\,(\omega^* - \omega) - 2\,p\omega^* + 2\,h\omega \tag{107}$$

get simplified to

$$v^*_{lmpq} - v_{lmhj} = (2\,h - 2\,p + q - j)\,\mathcal{M}^* + (2\,h - 2\,p)\,\omega^*, \tag{108}$$

[16] For justification of this operation, see Sect. 6 in Efroimsky and Williams (2009).

an expression containing both short-period contributions proportional to the mean anomaly, and long-period contributions proportional to the argument of the pericentre.

8.1.2 The secular, the purely-short-period, and the mixed-period parts of the torque

Now we see that the terms entering series (106) can be split into three groups:

(1) The terms with $p = h$ and $q = j$ constitute a secular part of the torque, as in such terms v_{lmhj} cancel with v^*_{lmpq}. This $\mathcal{M}$- and ω-independent part is furnished by

$$\overline{\mathcal{T}} = \sum_{l=2}^{\infty} 2\,GM_{sec}^2\,\frac{R^{2l+1}}{a^{2l+2}}\sum_{m=0}^{l}\frac{(l-m)!}{(l+m)!}\,m\,\sum_{p=0}^{l}F_{lmp}^2(i)\sum_{q=-\infty}^{\infty}G_{lpq}^2(e)\,k_l\,\sin\epsilon_{lmpq}\,. \tag{109}$$

(2) The terms with $p = h$ and $q \neq j$ constitute a purely short-period part of the torque:

$$\widetilde{\mathcal{T}} = -\sum_{l=2}^{\infty} 2\,GM_{sec}^2\,\frac{R^{2l+1}}{a^{2l+2}}\sum_{m=0}^{l}\frac{(l-m)!}{(l+m)!}\,m\,\sum_{p=0}^{l}F_{lmp}^2(i)$$
$$\sum_{q=-\infty}^{\infty}\sum_{\substack{j=-\infty\\ j\neq q}}^{\infty}G_{lpq}(e)\,G_{lpj}(e)\,k_l\,\sin\left[(q-j)\,\mathcal{M}-\epsilon_{lmpq}\right]. \tag{110}$$

(3) The terms with $p \neq h$, make a mixed-period part comprised of both short- and long-period terms:

$$\mathcal{T}^{mixed} = -\sum_{l=2}^{\infty} 2\,GM_{sec}^2\,\frac{R^{2l+1}}{a^{2l+2}}\sum_{m=0}^{l}\frac{(l-m)!}{(l+m)!}\,m\sum_{p=0}^{l}F_{lmp}(i)\sum_{\substack{h=0\\ h\neq p}}^{l}F_{lmh}(i)$$
$$\sum_{q=-\infty}^{\infty}\sum_{j=-\infty}^{\infty}G_{lhq}(e)\,G_{lpj}(e)\,k_l\,\sin\left[(2h-2p+q-j)\,\mathcal{M}^*\right.$$
$$\left.+(2h-2p)\omega^*-\epsilon_{lmpq}\right]. \tag{111}$$

8.1.3 The $l = 2$ and $l = 3$ terms in the $O(i^2)$ approximation

For $l = 2$, index m will take the values 0, 1, 2 only. Although the $m = 0$ terms enter the potential, they add nothing to the torque, because differentiation of (99) with respect to $-\theta$ furnishes the m multiplier in (106). To examine the remaining terms, we should consider the inclination functions with subscripts $(lmp) = (220), (210), (211)$ only:

$$F_{220}(i) = 3 + O(i^2),\; F_{210}(i) = \frac{3}{2}\,\sin i + O(i^2),\;\; F_{211}(i) = -\frac{3}{2}\,\sin i + O(i^2), \tag{112}$$

all the other $F_{2mp}(i)$ being of order $O(i^2)$ or higher. So for $p = h$ (i.e., both in the secular and short-period parts) it is sufficient, in the $O(i^2)$ approximation, to keep only the terms with $F^2_{220}(i)$, ignoring those with $F^2_{210}(i)$ and $F^2_{211}(i)$. We see that in the $O(i^2)$ approximation

– among the $l = 2$ terms, both in the secular and purely short-period parts, only the terms with $(lmp) = (220)$ are relevant.

For $p \neq h$, i.e., in the mixed-period part, the terms of the leading order in inclination are: $F_{lmp}(i)F_{lmh}(i) = F_{210}(i)F_{211}(i)$ and $F_{lmp}(i)F_{lmh}(i) = F_{211}(i)\,F_{210}(i)$, which happen to be equal to one another, and to be of order $O(i^2)$. This way, in the $O(i^2)$ approximation,

– the mixed-period part of the $l = 2$ component may be omitted.

The functions $F_{lmp} = F_{310}, F_{312}, F_{313}, F_{320}, F_{321}, F_{322}, F_{323}, F_{331}, \; F_{332}, F_{333}$ are of order $O(i)$ or higher. The terms containing the squares or cross-products of the these functions may thus be dropped. Specifically, the smallness of the cross-terms tells us that

– the mixed-period part of the $l = 3$ component may be omitted.

What remains is the terms containing the squares of functions

$$F_{311}(i) = -\frac{3}{2} + O(i^2) \qquad \text{and} \qquad F_{330}(i) = 15 + O(i^2). \tag{113}$$

In other words,

– among the $l = 3$ terms, both in the secular and purely short-period parts, only the terms with $(lmp) = (311)$ and $(lmp) = (330)$ are important in the $O(i^2)$ approximation.

All in all, for $l = 2, 3$, the mixed-period part of the torque may be dropped in the $O(i^2)$ approximation. The secular and the short-period parts will be developed up to e^6, inclusively.

8.2 Approximation for the secular and short-period parts of the tidal torque

As we saw, the secular and short-period parts of the torque may be approximated with

$$\begin{aligned}\overline{\mathcal{T}} &= \overline{\mathcal{T}}_{l=2} + \overline{\mathcal{T}}_{l=3} + O\left(\epsilon\,(R/a)^9\right)\\ &= \overline{\mathcal{T}}_{(lmp)=(220)} + \left[\overline{\mathcal{T}}_{(lmp)=(311)} + \overline{\mathcal{T}}_{(lmp)=(330)}\right] + O(\epsilon\, i^2) + O\left(\epsilon\,(R/a)^9\right),\end{aligned} \tag{114}$$

and

$$\begin{aligned}\widetilde{\mathcal{T}} &= \widetilde{\mathcal{T}}_{l=2} + \widetilde{\mathcal{T}}_{l=3} + O\left(\epsilon\,(R/a)^9\right)\\ &= \widetilde{\mathcal{T}}_{(lmp)=(220)} + \left[\widetilde{\mathcal{T}}_{(lmp)=(311)} + \widetilde{\mathcal{T}}_{(lmp)=(330)}\right] + O(\epsilon\, i^2) + O\left(\epsilon\,(R/a)^9\right),\end{aligned} \tag{115}$$

where the $l = 2$ and $l = 3$ inputs are of the order $(R/a)^5$ and $(R/a)^7$, accordingly, while the $l = 4, 5, \ldots$ inputs constitute $O(\epsilon\,(R/a)^9)$.

Expressions for $\overline{\mathcal{T}}_{(lmp)=(220)}$, $\overline{\mathcal{T}}_{(lmp)=(311)}$, and $\overline{\mathcal{T}}_{(lmp)=(310)}$ are presented in the Electronic Supplementary Material 3 "The $l = 2$ and $l = 3$ terms of the secular part of the torque". As an example, here we provide one of these components:

$$
\begin{aligned}
\overline{\mathcal{T}}_{(lmp)=(220)} \;=\;& \frac{3}{2}\, G M_{sec}^2\, R^5\, a^{-6} \left[\frac{1}{2304}\, e^6\, k_2 \sin|\epsilon_{220-3}|\, \operatorname{sgn}\,(-n - 2\,\dot{\theta}) \right.\\
&+ \left(\frac{1}{4}\, e^2 - \frac{1}{16}\, e^4 + \frac{13}{768}\, e^6 \right) k_2\, \sin|\epsilon_{220-1}|\, \operatorname{sgn}\,(n - 2\,\dot{\theta})\\
&+ \left(1 - 5\, e^2 + \frac{63}{8}\, e^4 - \frac{155}{36}\, e^6 \right) k_2\, \sin|\epsilon_{2200}|\, \operatorname{sgn}\,(n - \dot{\theta})\\
&+ \left(\frac{49}{4}\, e^2 - \frac{861}{16}\, e^4 + \frac{21975}{256}\, e^6 \right) k_2\, \sin|\epsilon_{2201}|\, \operatorname{sgn}\,(3\,n - 2\,\dot{\theta})\\
&+ \left(\frac{289}{4}\, e^4 - \frac{1955}{6}\, e^6 \right) k_2\, \sin|\epsilon_{2202}|\, \operatorname{sgn}\,(2\,n - \dot{\theta})\\
&\left. + \frac{714025}{2304}\, e^6\, k_2\, \sin|\epsilon_{2203}|\, \operatorname{sgn}\, \left(5n - 3\dot{\theta}\right) \right] + O(e^8\, \epsilon) + O(i^2\, \epsilon). \qquad (116)
\end{aligned}
$$

Here each term changes its sign on crossing the appropriate resonance. The change of the sign takes place smoothly, as the value of the term goes through zero—this can be seen from formula (91) and from the fact that the tidal mode ω_{lmpq} vanishes in the $lmpq$ resonance.

Expressions for $\widetilde{\mathcal{T}}_{(lmp)=(220)}$, $\widetilde{\mathcal{T}}_{(lmp)=(311)}$, and $\widetilde{\mathcal{T}}_{(lmp)=(330)}$ are given in the Electronic Supplementary Material 4 "The $l = 2$ and $l = 3$ terms of the short-period part of the torque". Although the average of the short-period part of the torque vanishes, this part does contribute to dissipation. Oscillating torques contribute also to variations of the surface of the tidally-distorted primary, the latter fact being of importance in laser-ranging experiments.

The modes $n(q-j)$ of the short-period torque are integers of n and thus are commensurate with the spin rate $\dot{\theta}$ near an A/B resonance, A and B being integer. It may be interesting to explore numerically the role of this torque when $q - j = 1$ and $A/B = N$ is integer. The possible role of the short-period torque in the entrapment and libration dynamics has never been discussed so far, as the previous studies employed expressions for the tidal torque, which were obtained through averaging over the period of the secondary's orbital motion.

9 Marking the minefield

Our expressions for the secular and purely short-period parts of the tidal torque look cumbersome when compared to the compact and simple formulae employed in the literature hitherto. It will therefore be important to explain why those simplifications are impractical.

9.1 Perils of the conventional simplification

Insofar as the quality factor is large and the lag is small (i.e., insofar as $\sin\epsilon$ can be approximated with ϵ), the $\{lmp\} = \{220\}$ part of the torque assumes a simpler form:

$$
{}^{(Q>10)}\overline{\mathcal{T}}_{l=2} = \frac{3}{2}\, G M_{sec}^2 R^5 a^{-6} k_2 \sum_{q=-3}^{3} G_{20q}^{\;2}(e)\, \epsilon_{220q} + O(e^6\, \epsilon) + O(i^2\, \epsilon) + O(\epsilon^3), \qquad (117)
$$

where the error $O(\epsilon^3)$ originates from $\sin\epsilon = \epsilon + O(\epsilon^3)$.

The simplification conventionally used in the literature ignores the frequency-dependence of the Love number and attributes the overall frequency-dependence to the lag. It also ignores the difference between the tidal lag ϵ and the lag in the material, δ. This way, the conventional

simplification makes the tidal lag ϵ obey the scaling law (75b). At this point, most authors also set $\alpha = -1$. Here we shall explore this approach, though shall keep α arbitrary. From the formula [17]

$$\Delta t_{lmpq} = \mathcal{E}^{-\alpha}\ \chi_{lmpq}^{-(\alpha+1)} \tag{118}$$

derived by Efroimsky and Lainey (2007) in the said approach, we see that the time lags are related to the principal-frequency lag Δt_{2200} via:

$$\Delta t_{lmpq} = \Delta t_{2200}\left(\frac{\chi_{2200}}{\chi_{lmpq}}\right)^{\alpha+1}. \tag{119}$$

When the despinning is still going on and $\dot\theta \gg n$, the corresponding phase lags are:

$$\epsilon_{lmpq} \equiv \Delta t_{lmpq}\,\omega_{lmpq} = -\Delta t_{2200}\,\chi_{lmpq}\left(\frac{\chi_{2200}}{\chi_{lmpq}}\right)^{\alpha+1} = -\epsilon_{2200}\left(\frac{\chi_{2200}}{\chi_{lmpq}}\right)^{\alpha}, \tag{120}$$

which helps us to cast the secular part of the torque into the following convenient form[18]:

$$\begin{aligned} {}^{(Q>10)}\overline{\mathcal{T}}_{l=2} &= \mathcal{Z}\left[-\dot\theta\left(1+\frac{15}{2}e^2+\frac{105}{4}e^4+O(e^6)\right)+\ n\left(1+\left(\frac{15}{2}-6\alpha\right)e^2\right.\right.\\ &\quad\left.\left.+\left(\frac{105}{4}-\frac{363}{8}\alpha\right)e^4+O(e^6)\right)\right]+O(i^2/Q)+O(Q^{-3})+O(\alpha e^2 Q^{-1}n/\dot\theta) \end{aligned} \tag{121a}$$

$$\approx \mathcal{Z}\left[-\dot\theta\left(1+\frac{15}{2}e^2\right)+n\left(1+\left(\frac{15}{2}-6\,\alpha\right)e^2\right)\right], \tag{121b}$$

where the overall factor reads as:

$$\begin{aligned}\mathcal{Z} = \frac{3GM_{sec}^2 k_2\Delta t_{2200}R^5}{a^6} &= \frac{3n^2M_{sec}^2k_2\Delta t_{2200}R^5}{(M_{prim}+M_{sec})a^3}\\ &= \frac{3\,n\,M_{sec}^2\,k_2\,R^5}{Q_{2200}(M_{prim}+M_{sec})a^3}\,\frac{n}{\chi_{2200}},\end{aligned} \tag{122}$$

---

[17] Let $\frac{1}{\sin\epsilon} = (\mathcal{E}\chi)^{\alpha}$, where $\mathcal{E}$ is an empirical parameter of the dimensions of time, while ϵ is small enough, so $\sin\epsilon \approx \epsilon$. In combination with $\epsilon_{lmpq} \equiv \omega_{lmpq}\,\Delta t_{lmpq}$ and $\chi_{lmpq} = |\omega_{lmpq}|$, this yields (118).

[18] For $\dot\theta \gg 2n$, all the modes ω_{220q} are negative, so $\omega_{220q} = -\chi_{220q}$. Then, keeping in mind that $n/\dot\theta \ll 1$, we process (120), for $q=1$, as

$$\begin{aligned}\Delta t_{2200}\,\chi_{2200}\left(\frac{\chi_{2200}}{\chi_{2201}}\right)^{\alpha} &= -\Delta t_{2200}\,2|n-\dot\theta|\left(\frac{2|n-\dot\theta|}{|-2\dot\theta+3n|}\right)^{\alpha}\\ &= -\Delta t_{2200}\,2(\dot\theta-n)\left[1+\frac{\alpha}{2}\frac{n}{\dot\theta}+O\left((n/\dot\theta)^2\right)\right]\end{aligned}$$

and similarly for the other values of q, and then plug the results into (117).

M_{prim} and M_{sec} being the primary's and secondary's masses.[19] Dividing (122) by the primary's principal moment of inertia $\xi\; M_{primary}\; R^2$, we obtain the contribution that this component of the torque brings into the angular deceleration rate $\ddot{\theta}$:

$$\ddot{\theta} \;=\; \mathcal{K}\,\left\{-\,\dot{\theta}\,\left[\,1+\frac{15}{2}\,e^2+\frac{105}{4}\,e^4+O(e^6)\,\right]+n\left[1+\left(\frac{15}{2}-6\alpha\right)e^2 \right.\right.$$
$$\left.\left.+\;\left(\frac{105}{4}-\frac{363}{8}\alpha\right)e^4+O(e^6)\right]\right\}+O(i^2/Q)+O(Q^{-3})+O(\alpha e^2 Q^{-1} n/\dot{\theta}) \qquad \text{(123a)}$$

$$\approx\;\mathcal{K}\,\left[-\,\dot{\theta}\,\left(1+\frac{15}{2}\,e^2\right)+\,n\,\left(1+\left(\frac{15}{2}-6\,\alpha\right)\,e^2\right)\right], \qquad \text{(123b)}$$

the factor $\mathcal{K}$ being given by

$$\mathcal{K} \equiv \frac{\mathcal{Z}}{\xi\; M_{prim}\; R^2} = \frac{3\,n^2\; M_{sec}^2\;\, k_2\;\, \Delta t_{2200}}{\xi\;\, M_{prim}\;\, (M_{prim}+M_{sec})}\;\frac{R^3}{a^3}$$
$$= \frac{3\,n\; M_{sec}^2\;\, k_2}{\xi\;\, Q_{2200}\;\; M_{prim}\;\, (M_{prim}+M_{sec})}\;\frac{R^3}{a^3}\;\;\frac{n}{\chi_{2200}}\,, \qquad \text{(124)}$$

where ξ is a multiplier emerging in the expression $\xi\; M_{primary}\; R^2$ for the primary's principal moment of inertia ($\xi = 2/5$ for a homogeneous sphere).

In the special case of $\alpha = -1$, the above expressions enjoy agreement with the appropriate result stemming from the corrected MacDonald model—except that our (121–123) contain Δt_{2200}, Q_{2200}, χ_{2200} instead of Δt, Q, χ standing in formulae (44–47) from Williams and Efroimsky (2012). Formula (123b) says that the secular part of the tidal torque vanishes for

$$\dot{\theta} - n = -6\; n\; e^2\; \alpha. \qquad \text{(125)}$$

For $\alpha = -1$, this coincides with the result obtained in Rodríguez et al. (2008, eqn. 2.4), Correia et al. (2011, eqn. 20), and Williams and Efroimsky (2012, eqn. 49). The coincidence however should not be taken at its face value, because it is *occasional* or, possibly better to say, *exceptional*. Formulae (121–122) were obtained by insertion of the expressions for the eccentricity functions and the phase lags into (117), and *by assuming that* $n \ll |\dot{\theta}|$. The latter caveat is a crucial element, not to be overlooked by the users of formulae (121–122) and of their corollary (123–124) for the tidal deceleration rate.

The case of $\alpha = -1$ is special, in that it permits derivation of (121–125) *without assuming that* $n \ll |\dot{\theta}|$. However for $\alpha > -1$ the condition $n \ll |\dot{\theta}|$ remains mandatory, so formulae (121–124) become *inapplicable* when $\dot{\theta}$ reduces to values of about several n.

Although formulae (121a) and (123a) contain an absolute error $O(\alpha e^2 Q^{-1} n/\dot{\theta})$, this does *not* mean that for $\dot{\theta}$ close to n the absolute error becomes $O(\alpha e^2 Q^{-1})$ and the relative one becomes $O(\alpha e^2)$. In reality, for $\dot{\theta}$ comparable to n, *the entire approximation falls apart*, because formulae (119–120) were derived from expression (118), which is valid for $Q \gg 1$

[19] To arrive at the right-hand side of (122), we recalled that $\chi_{lmpq}\;\Delta t_{lmpq} = |\epsilon_{lmpq}|$ and that $Q^{-1}_{lmpq} = |\epsilon_{lmpq}| + O(\epsilon^3) = |\epsilon_{lmpq}| + O(Q^{-3})$, according to formula (12).

only (unless $\alpha = -1$). So these formulae become inapplicable in the vicinity of a commensurability. By ignoring this limitation, one can easily encounter unphysical paradoxes.[20]

Thence, in all situations, except for the unrealistic rheology $\alpha = -1$, limitations of the approximation (121–124) should be kept in mind. This approximation remains acceptable for $n \ll |\dot{\theta}|$, but becomes misleading on approach to the physically-interesting resonances.

9.2 An oversight in a popular formula

The form in which our approximation (121–124) is cast may appear awkward. Specifically, expression (124) for the despinning rate $\ddot{\theta}$ is written as a function of $\dot{\theta}$ and n, multiplied by the factor $\mathcal{K}$. This form would be reasonable, were $\mathcal{K}$ a constant. That it is not—can be seen from the presence of the multiplier $n/\chi_{2200} = \dfrac{n}{2\,|\,\dot{\theta}\, - n|}$ on the right-hand side of (124).

Still, when written in this form, our result is easy to juxtapose with an appropriate formula from Correia and Laskar (2004, 2009). There, the expression for the despinning rate looks similar to ours, up to an important detail: the overall factor is a constant, because it lacks the said multiplier $\frac{n}{\chi_{2200}}$. The multiplier was lost in those two papers, because the quality factor was introduced there as $1/(n\,\Delta t)$, see the line after formula (9) in Correia and Laskar (2009). In reality, the quality factor Q should, of course, be a function of the tidal frequency χ, because Q serves the purpose of describing the tidal damping at this frequency. Had the quality factor been taken as $1/(\chi\,\Delta t)$, it would render the corrected MacDonald model ($\alpha = -1$), and the missing multiplied would be there. Being unphysical,[21] the model is mathematically convenient, because it enables one to write down the secular part of the torque in a compact form, avoiding the expansion into a series (Williams and Efroimsky 2012). The model was pioneered by Singer (1968) and employed by Mignard (1979, 1980); Hut (1981) and other authors.

Interestingly, in the special case of the 3:2 spin–orbit resonance, we have $\chi = n$. Still, the difference between χ and n in the vicinity of the resonance may alter the probability of entrapment of Mercury into this rotation mode. The difference between χ and n becomes even more considerable near the other resonances of interest. So the probabilities of entrapment into those resonances must be recalculated.

10 Conclusions

The goal of this paper was to lay the ground for a model of tidal entrapment into spin–orbital resonances. To this end, we approached the tidal theory from the first principles of solid-state mechanics. Starting from the expression for the material's compliance in the time domain, we derived the frequency-dependence of the Fourier components of the tidal torque. The other torque, one caused by the triaxiality of the rotator, is not a part of this study and will be addressed elsewhere.

– We base our work on the Andrade rheological model, providing arguments in favour of its applicability to the Earth's mantle, and therefore, very likely, to other terrestrial planets and moons. The model is also likely to apply to the icy moons (Castillo-Rogez et al.

[20] For example, in the case of $\alpha > -1$, formulae (118–120) render infinite values for Δt_{lmpq} and ϵ_{lmpq} on crossing the commensurability, i.e., when ω_{lmpq} goes through zero.

[21] To be exact, the model is unphysical everywhere except in the closest vicinity of the resonance—see formulae (88–90).

2011). We have reformulated the model in terms of a characteristic anelastic timescale $\tau_{_A}$ (the Andrade time). The ratio of the Andrade time to the viscoelastic Maxwell time, $\zeta = \tau_{_A}/\tau_{_M}$, serves as a dimensionless free parameter of the rheological model.
The parameters $\tau_{_A}$, $\tau_{_M}$, ζ cannot be regarded constant, though their values may be changing very slowly over vast bands of frequency. The shapes of these frequency-dependencies may depend on the dominating dissipation mechanisms and, thereby, on the magnitude of the load, as different damping mechanisms get activated under different loads.
The main question here is whether, in the low-frequency limit, anelasticity becomes much weaker than viscosity. (That would imply an increase of $\tau_{_A}$ and ζ as the tidal frequency χ goes down.) The study of ices under weak loads, with friction caused mainly by lattice diffusion (Castillo-Rogez et al. 2011; Castillo-Rogez and Choukroun 2010) has not shown such a decline of anelasticity. However, Karato and Spetzler (1990) point out that it should be happening in the Earth's mantle, where the loads are much higher and damping is caused mainly by unpinning of dislocations. According to Karato and Spetzler (1990), in the Earth, the decrease of the role of anelasticity happens abruptly as the frequency falls below the threshold $\chi_0 \sim 1\,\text{year}^{-1}$. We then may expect a similar switch in the other terrestrial planets and the Moon, though there the threshold may be different as it is sensitive to the temperature of the mantle. The question, though, remains if this statement is valid also for the small bodies, in which the tidal stresses are weaker and dissipation may be dominated by lattice diffusion.

– We have derived the frequency dependencies of the factors $k_l\,\sin\epsilon_l$ emerging in the tidal theory. Naturally, the obtained dependencies of these factors upon the tidal frequency χ_{lmpq} (or, more exactly, upon the tidal mode ω_{lmpq}) mimic the frequency-dependence of the imaginary part of the complex compliance. They scale as $\sim\chi^{-\alpha}$ with $0<\alpha<1$, at higher frequencies; and as $\sim\chi^{-1}$ at lower frequencies. However in the zero-frequency limit the factors $k_l\,\sin\epsilon_l$ demonstrate a behaviour inherent in the tidal lagging and absent in the case of lagging in a sample: in a close vicinity of the zero frequency, these factors (and the appropriate components of the tidal torque) become linear in the frequency. This way, $k_l\,\sin\epsilon_l$ first reaches a finite maximum, then decreases continuously to nil as the frequency approaches to zero, and then changes its sign. So the resonances are crossed continuously, with neither the tidal torque nor the tidal force diverging there. For example, the leading term of the torque vanishes at the synchronous orbit.
The continuous traversing of resonances was pointed out in a heuristic manner by Efroimsky and Williams (2009). Now we have derived this result directly from the expression for the compliance of the material of the rotating body. Our treatment however has a flaw: the frequency, below which the factors $k_l\,\sin\epsilon_l$ and the appropriate components of the torque change their frequency-dependence to linear, is very low (lower than 10^{-10} Hz, if we take our formulae literally). The reason for this is that we kept using the known value of the Maxwell time $\tau_{_M}$ all the way down to the zero frequency. Possible changes of the Maxwell time in the zero-frequency limit may broaden the region of linear dependence.

– We have offered an explanation of the "improper" frequency-dependence of dissipation in the Moon, discovered by LLR. The main point of our explanation is that the LLR measures the *tidal* dissipation whose frequency-dependence is different from that of the *seismic* dissipation. Specifically, the "wrong" sign of the exponential in the power dissipation law may indicate that the frequencies at which tidal friction was observed were below the frequency at which the lunar $k_2\,\sin\epsilon_2$ has its peak. Taken the relatively high frequencies of observation (corresponding to periods of order month to year), this explanation can be

accepted only if the lunar mantle has a low mean viscosity. This may be the case, taken the presumably high concentration of the partial melt in the low mantle.

– We have developed a detailed formalism for the tidal torque, and have singled out its oscillating component.
  The studies of entrapment into spin–orbit resonances, performed in the past, took into account neither the afore-mentioned complicated frequency-dependence of the torque in the vicinity of a resonance, nor the oscillating part of the torque.

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Appendix

A Interconnection between the quality factor and the phase lag

The power P exerted by a tide-raising secondary on its primary can be written as

$$P = -\int \rho \ \mathbf{V} \ \cdot \ \nabla W \ d^3x \tag{126}$$

ρ, $\mathbf{V}$, and W signifying the density, velocity, and tidal potential in the small volume d^3x of the primary. Since $\nabla \cdot (\rho\,\mathbf{V}) + \frac{\partial \rho}{\partial t} = 0$, the dot product takes the form of

$$\rho\ \mathbf{V} \cdot \nabla W = \nabla \cdot (\rho\ \mathbf{V}\ W) - \rho\ W\ \nabla \cdot\ \mathbf{V} - \ \mathbf{V}\ W\ \nabla \rho. \tag{127}$$

Under the realistic assumption of the primary's incompressibility, the term with $\nabla \cdot \mathbf{V}$ may be omitted. To get rid of the term with $\nabla \rho$, one has to accept a much stronger approximation of the primary being homogeneous. Then the power will be rendered by

$$P = -\int \nabla \ \cdot \ (\rho \ \mathbf{V} \ W)\, d^3x = -\int \rho \ W \ \mathbf{V} \ \cdot \ \mathbf{n} \ dS, \tag{128}$$

$\mathbf{n}$ being the outward normal and dS being an element of the surface area of the primary. This expression for the power (pioneered, probably, by Goldreich 1963) enables one to calculate the work through radial displacements only, in neglect of horizontal motion.

Denoting the radial elevation with ζ, we write the power per unit mass, $\mathcal{P} \equiv P/M$, as:

$$\mathcal{P} = \left(-\frac{\partial W}{\partial r}\right)\ \mathbf{V} \cdot \mathbf{n} = \left(-\frac{\partial W}{\partial r}\right) \frac{d\zeta}{dt}. \tag{129}$$

A harmonic external potential

$$W = W_0 \ \cos(\omega_{lmpq}\, t), \tag{130}$$

applied at a point of the primary's surface, will elevate this point by

$$\zeta = h_2\,\frac{W_o}{\mathrm{g}}\,\cos\left(\omega_{lmpq}\,t - \epsilon_{lmpq}\right) = h_2\,\frac{W_o}{\mathrm{g}}\,\cos\left(\omega_{lmpq}\,t - \omega_{lmpq}\,\Delta t_{lmpq}\right), \tag{131}$$

with g being the surface gravity acceleration, and h_2 denoting the Love number.

In formula (131), ω_{lmpq} is one of the modes (101) showing up in the Darwin–Kaula expansion (99). The quantity $\epsilon_{lmpq} = \omega_{lmpq}\,\Delta t_{lmpq}$ is the corresponding phase lag, while Δt_{lmpq} is the positively defined time lag at this mode. Although the tidal modes ω_{lmpq} can assume any sign, both the potential W and elevation ζ can be expressed via the positively defined forcing frequency $\chi_{lmpq} = |\,\omega_{lmpq}\,|$ and the absolute value of the phase lag:

$$W = W_0\,\cos(\chi\,t), \tag{132}$$

$$\zeta = h_2\,\frac{W_o}{\mathrm{g}}\,\cos(\chi\,t - |\,\epsilon\,|), \tag{133}$$

subscripts $lmpq$ being dropped here and hereafter for brevity.

The vertical velocity of the considered element of the primary's surface will be

$$\frac{d\zeta}{dt} = -h_2\chi\,\frac{W_o}{\mathrm{g}}\,\sin(\chi t - |\epsilon|) = -h_2\chi\,\frac{W_o}{\mathrm{g}}\,(\sin\chi t\cos|\epsilon| - \cos\chi t\sin|\epsilon|)\,. \tag{134}$$

Introducing the notation $A = h_2\,\frac{W_0}{\mathrm{g}}\,\frac{\partial W_0}{\partial r}$, we write the power per unit mass as

$$\mathcal{P} = \left(-\frac{\partial W}{\partial r}\right)\frac{d\zeta}{dt} = A\,\chi\,\cos(\chi\,t)\,\sin(\chi t - |\epsilon|), \tag{135}$$

and express the work w per unit mass, performed over a time interval $(t_0,\ t)$, as:

$$\begin{aligned} w\big|_{t_0}^{t} &= \int_{t_0}^{t}\mathcal{P}\,dt = A\int_{\chi t_0}^{\chi t}\cos(\chi\,t)\,\sin(\chi t - |\epsilon|)d(\chi\,t) \\ &= A\,\cos|\epsilon|\int_{\chi t_0}^{\chi t}\cos z\,\sin z\,dz - A\,\sin|\epsilon|\int_{\chi t_0}^{\chi t}\cos^2 z\,dz \\ &= -\frac{A}{4}\,[\cos(2\chi t - |\epsilon|) + 2\,\chi\,t\,\sin|\epsilon|\,]_{t_0}^{t}. \end{aligned} \tag{136}$$

Being cyclic, the first term in (136) renders the elastic energy stored in the body. The second term, being linear in time, furnishes the energy damped. This clear interpretation of the two terms was offered by Stan Peale (2011, personal communication).

The work over a time period $T = 2\pi/\chi$ is equal to the energy dissipated over the period:

$$w\big|_{t=0}^{t=T} = \Delta E_{cycle} = -A\,\pi\,\sin|\epsilon|. \tag{137}$$

The peak *work* is obtained over the time span from π to $|\epsilon|$ and assumes the value

$$E_{peak}^{(work)} = \frac{A}{2}\left[\cos|\epsilon| - \sin|\epsilon|\left(\frac{\pi}{2} - |\epsilon|\right)\right], \tag{138}$$

whence the appropriate quality factor is given by:

$$Q_{work}^{-1} = \frac{-\Delta E_{cycle}}{2\,\pi\,E_{peak}^{(work)}} = \frac{\tan|\epsilon|}{1 - \left(\frac{\pi}{2} - |\epsilon|\right)\,\tan|\epsilon|}. \tag{139}$$

To calculate the peak *energy* stored in the body, we would note that the first term in (136) is maximal when taken over the span from $\chi\,t = \pi/4 + |\epsilon|/2$ through $\chi\,t = 3\pi/4 + |\epsilon|/2$:

$$E_{peak}^{(energy)} = \frac{A}{2}, \tag{140}$$

and the corresponding quality factor is:

$$Q^{-1}_{energy} = \frac{-\Delta E_{cycle}}{2\pi\; E^{(energy)}_{peak}} = \sin|\epsilon|. \tag{141}$$

Goldreich (1963) suggested to employ the span $\chi\, t = (0,\;\pi/4)$. The absolute value of the resulting power, denoted by Goldreich as E^{*}, is equal to

$$E^{*} = \frac{A}{2}\,\cos|\epsilon| \tag{142}$$

and is neither the peak value of the energy stored, nor that of the work performed. Goldreich (1963) however employed it to define a quality factor, which we shall term $Q_{Goldreich}$:

$$Q^{-1}_{Goldreich} = \frac{-\Delta E_{cycle}}{2\pi\; E^{*}} = \tan|\epsilon|. \tag{143}$$

The quality factor Q_{energy} defined through (141) is preferable, as the expansion of tides contains terms proportional to $k_l(\chi_{lmpq})\,\sin\epsilon_l(\chi_{lmpq})$. Since the long-established tradition suggests to substitute $\sin|\epsilon|$ with $1/Q$, it is advisable to define the Q exactly as (141), and also to call it Q_l, to distinguish it from the seismic quality factor (Efroimsky 2012).

B The correspondence principle (elastic-viscoelastic analogy)

B.1 The correspondence principle, for nonrotating bodies

While the static Love numbers depend on the static rigidity μ through (3), it is not immediately clear if a similar formula interconnects also $\bar{k}_l(\chi)$ with $\bar{\mu}(\chi)$. To understand why and when the relation should hold, recall that formulae (3) originate from the solution of a boundary-value problem for a system incorporating two equations:

$$\begin{aligned} \sigma_{\beta\nu} &= 2\mu\; u_{\beta\nu}, \\ 0 &= \frac{\partial\sigma_{\beta\nu}}{\partial x_\nu} - \frac{\partial p}{\partial x_\beta} - \rho\,\frac{\partial(W+U)}{\partial x_\beta}, \end{aligned} \tag{144a}$$

the latter being simply the equation of equilibrium written for a *static* viscoelastic medium, in neglect of compressibility and heat conductivity. The notations $\sigma_{\beta\nu}$ and $u_{\beta\nu}$ stand for the *deviatoric* stress and strain, $p \equiv -\frac{1}{3}\,\mathrm{Sp}\,\mathbb{S}$ is the pressure (set to be nil in incompressible media), while W and U are the perturbing and perturbed potentials. By solving the system, one arrives at the static relation $U_l = k_l\,W_l$, with the customary static Love numbers k_l expressed via ρ, R, and μ by (3).

Generalise (144a–144a) to the case of time-dependent loading of a *nonrotating* body:

$$\mathbb{S} = 2\,\hat{\mu}\,\mathbb{U}, \tag{145a}$$

$$\rho\,\ddot{\mathbf{u}} = \nabla\mathbb{S} - \nabla p - \rho\,\nabla(W+U) \tag{145b}$$

or, in terms of components:

$$\sigma_{\beta\nu} = 2\hat{\mu}\,u_{\beta\nu}, \tag{146a}$$

$$\rho\,\ddot{u}_\beta = \frac{\partial\sigma_{\beta\nu}}{\partial x_\nu} - \frac{\partial p}{\partial x_\beta} - \rho\,\frac{\partial(W+U)}{\partial x_\beta}. \tag{146b}$$

In the frequency domain, this will look:

$$\bar{\sigma}_{\beta\nu}(\chi) = 2\,\bar{\mu}(\chi)\,\bar{u}_{\beta\nu}(\chi), \tag{147a}$$

$$\rho\,\chi^2\,\bar{u}_{\beta\nu}(\chi) = \frac{\partial\bar{\sigma}_{\beta\nu}(\chi)}{\partial x_\nu} - \frac{\partial\bar{p}(\chi)}{\partial x_\beta} - \rho\,\frac{\partial\left[\bar{W}(\chi) + \bar{U}(\chi)\right]}{\partial x_\beta}, \tag{147b}$$

where a bar denotes a spectral component for all functions except μ—recall that $\bar{\mu}$ is a spectral component not of the kernel $\mu(\tau)$ but of its time-derivative $\dot{\mu}(\tau)$.

Unless the frequencies are extremely high, we can neglect the body-fixed acceleration term $\chi^2\,\bar{u}_{\beta\nu}(\chi)$ in the second equation, in which case our system of equations for the spectral components will mimic (144). Thus we arrive at the so-called *correspondence principle* (also known as the *elastic-viscoelastic analogy*), which maps a solution of a linear viscoelastic boundary-value problem to a solution of a corresponding elastic problem with the same initial and boundary conditions. As a result, the algebraic equations for the Fourier (or Laplace) components of the strain and stress in the viscoelastic case mimic the equations connecting the strain and stress in the appropriate elastic problem. So the viscoelastic operational moduli $\bar{\mu}(\chi)$ or $\bar{J}(\chi)$ obey the same algebraic relations as the elastic parameters μ or J.

In the literature, there is no consensus on the authorship of this principle. For example, Haddad (1995) mistakenly attributes it to several authors who published in the 1950s and 1960s. In reality, the principle was pioneered almost a century earlier by Darwin (1879), for isotropic incompressible media. The principle was extended to more general types of media by Biot (1954, 1958), who also pointed out some limitations on its applicability.

B.2 The correspondence principle, for rotating bodies

Consider a body of mass M_{prim}, which is spinning at a rate $\boldsymbol{\omega}$ and is also performing some orbital motion (for example, is orbiting, with its partner of mass M_{sec}, around their mutual centre of mass). Relative to some inertial coordinate system, the centre of mass of the body is located at $\mathbf{x}_{CM}$, while a small parcel of its material is positioned at $\mathbf{x}$. Relative to the centre of mass of the body, the parcel is located at $\mathbf{r} = \mathbf{x} - \mathbf{x}_{CM}$. The body being deformable, we can decompose $\mathbf{r}$ into its average value, $\mathbf{r}_0$, and an instantaneous displacement $\mathbf{u}$:

$$\left.\begin{array}{r}\mathbf{x} = \mathbf{x}_{CM} + \mathbf{r} \\ \mathbf{r} = \mathbf{r}_0 + \mathbf{u}\end{array}\right\} \Longrightarrow \mathbf{x} = \mathbf{x}_{CM} + \mathbf{r}_0 + \mathbf{u}. \tag{148}$$

Denote with D/Dt the time-derivative in the inertial frame. The symbol d/dt and overdot will be reserved for the time-derivative in the body frame, so $d\mathbf{r}_0/dt = 0$. Then

$$\frac{D\mathbf{r}}{Dt} = \frac{d\mathbf{r}}{dt} + \boldsymbol{\omega}\times\mathbf{r} \quad\text{and}\quad \frac{D^2\mathbf{r}}{Dt^2} = \frac{d^2\mathbf{r}}{dt^2} + 2\boldsymbol{\omega}\times\frac{d\mathbf{r}}{dt} + \boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r}) + \dot{\boldsymbol{\omega}}\times\mathbf{r}. \tag{149}$$

Together, the above formulae result in

$$\begin{aligned}\frac{D^2\mathbf{x}}{Dt^2} &= \frac{D^2\mathbf{x}_{CM}}{Dt^2} + \frac{D^2\mathbf{r}}{Dt^2} = \frac{D^2\mathbf{x}_{CM}}{Dt^2} + \frac{d^2\mathbf{r}}{dt^2} + 2\boldsymbol{\omega}\times\frac{d\mathbf{r}}{dt} + \boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r}) + \dot{\boldsymbol{\omega}}\times\mathbf{r} \\ &= \frac{D^2\mathbf{x}_{CM}}{Dt^2} + \frac{d^2\mathbf{u}}{dt^2} + 2\boldsymbol{\omega}\times\frac{d\mathbf{u}}{dt} + \boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r}) + \dot{\boldsymbol{\omega}}\times\mathbf{r}.\end{aligned} \tag{150}$$

The equation of motion for a small parcel of the body's material will read as

$$\rho\,\frac{D^2\mathbf{x}}{Dt^2} = \nabla\mathbb{S} - \nabla p + \mathbf{F}_{self} + \mathbf{F}_{ext}, \tag{151}$$

where $\mathbf{F}_{ext}$ is the exterior gravity force *per unit volume*, while $\mathbf{F}_{self}$ is the "interior" gravity force *per unit volume*, i.e., the self-force wherewith the rest of the body is acting upon the selected parcel of medium. Insertion of (150) in (151) furnishes:

$$\rho\left[\frac{D^2\mathbf{x}_{CM}}{Dt^2}+\mathbb{U}+2\boldsymbol{\omega}\times\,\dot{\mathbf{u}}+\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r})+\dot{\boldsymbol{\omega}}\times\mathbf{r}\right]=\nabla\mathbb{S}-\nabla p+\mathbf{F}_{self}+\mathbf{F}_{ext}. \quad (152)$$

At the same time, for the primary body as a whole, we can write:

$$M_{prim}\,\frac{D^2\mathbf{x}_{CM}}{Dt^2}=\int_V\mathbf{F}_{ext}\;d^3\mathbf{r}, \quad (153)$$

the integration being carried out over the volume V of the primary. (Recall that $\mathbf{F}_{ext}$ is a force per unit volume.) Combined together, the above two equations will result in

$$\rho\left[\mathbb{U}+2\,\boldsymbol{\omega}\times\,\dot{\mathbf{u}}+\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r})+\dot{\boldsymbol{\omega}}\times\mathbf{r}\right]$$
$$=\nabla\mathbb{S}-\nabla p+\mathbf{F}_{self}+\mathbf{F}_{ext}-\frac{\rho}{M_{prim}}\int_V\mathbf{F}_{ext}\;d^3\mathbf{r}. \quad (154)$$

For a spherically-symmetrical (not necessarily radially-homogeneous) body, the integral on the right-hand side removes the Newtonian part of the force, leaving the harmonics intact:

$$\mathbf{F}_{ext}-\frac{\rho}{M_{prim}}\int_V\mathbf{F}_{ext}\,d^3\mathbf{r}=\rho\sum_{l=2}^{\infty}\nabla W_l, \quad (155)$$

where the harmonics are given by

$$W_l(\mathbf{r},\;\mathbf{r}^*)=-\frac{GM_{sec}}{r^*}\;\left(\frac{r}{r^*}\right)^l\;P_l(\cos\gamma), \quad (156)$$

$\mathbf{r}^*$ being the vector pointing from the centre of mass of the primary to that of the secondary, and γ being the angle between $\mathbf{r}$ and $\mathbf{r}^*$, subtended at the centre of mass of the primary.

In reality, a tiny extra force $\mathcal{F}$, the tidal force per unit volume, is left over due to the body being slightly distorted:

$$\mathbf{F}_{ext}-\frac{\rho}{M_{prim}}\int_V\mathbf{F}_{ext}\,d^3\mathbf{r}=\rho\sum_{l=2}^{\infty}\nabla W_l+\mathcal{F}, \quad (157)$$

where $\mathcal{F}$ is the density multiplied by the average tidal acceleration experienced by the body as a whole. In neglect of $\mathcal{F}$, we arrive at

$$\rho\left[\mathbb{U}+\;2\,\boldsymbol{\omega}\times\,\dot{\mathbf{u}}+\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r})+\mathbb{U}\times\mathbf{r}\right]=\nabla\mathbb{S}-\nabla p-\rho\sum_{l=2}^{\infty}\nabla(U_l+W_l). \quad (158)$$

Here, to each disturbing term of the exterior potential, W_l, corresponds a term U_l of the self-potential, the self-force thus being expanded into $\mathbf{F}_{self}=-\sum_{l=2}^{\infty}\nabla U_l$.

Equation (158) could as well have been derived in the body frame.

Denoting the tidal frequency with χ, we see that the terms on the left-hand side have the order of $\rho\,\chi^2\,u$, $\rho\,\omega\,\chi\,u$, $\rho\,\omega^2\,r$, and $\rho\,\dot{\chi}\,\omega\,r$, correspondingly. In realistic situations, the first two terms, thus, can be neglected, and we end up with

$$0 = \nabla \mathbb{S} - \nabla p - \rho \sum_{l=2}^{\infty} \nabla (U_l + W_l) - \rho\, \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) - \rho\, \dot{\boldsymbol{\omega}} \times \mathbf{r}, \tag{159}$$

the term $-\nabla p$ vanishing in an incompressible media.

B.3 The centripetal term and the zero-degree Love number

The centripetal term in (159) can be split into a purely radial part and a part that can be incorporated into the W_2 term of the tide-raising potential, as was suggested by Love (1909, 1911). Introducing the colatitude ϕ' through $\cos\phi' = \frac{\boldsymbol{\omega}}{|\boldsymbol{\omega}|} \cdot \frac{\mathbf{r}}{|\mathbf{r}|}$, we can write:

$$\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) = \boldsymbol{\omega}\, (\boldsymbol{\omega} \cdot \mathbf{r}) - \mathbf{r}\, \omega^2 = \nabla \left[\frac{1}{2}\, (\boldsymbol{\omega} \cdot \mathbf{r})^2 - \frac{1}{2}\, \omega^2\, \mathbf{r}^2 \right] = \nabla \left[\frac{1}{2}\, \omega^2\, \mathbf{r}^2\, \left(\cos^2\phi' - 1 \right) \right].$$

The definition $P_2(\cos\phi') = \frac{1}{2}\left(3\,\cos^2\phi' - 1\right)$ renders: $\cos^2\phi' = \frac{2}{3}\, P_2(\cos\phi') + \frac{1}{3}$, whence:

$$\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) = \nabla \left[\frac{1}{3}\, \omega^2\, \mathbf{r}^2\, \left[P_2\left(\cos\phi'\right) - 1 \right] \right]. \tag{160}$$

We see that the centripetal force splits into a second-harmonic and purely-radial parts:

$$-\,\rho\, \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) = -\nabla \left[\frac{\rho}{3}\, \omega^2\, \mathbf{r}^2\, P_2\left(\cos\phi'\right) \right] + \nabla\, \left[\frac{\rho}{3}\, \omega^2\, \mathbf{r}^2 \right], \tag{161}$$

where we assume the body to be homogeneous. The second-harmonic part can be incorporated into the external potential. The response to this part will be proportional to the degree-2 Love number k_2.

The purely radial part of the centripetal potential generates a radial deformation. This part of the potential is often ignored, the associated deformation being tacitly included into the equilibrium shape of the body. Compared to the main terms of the equation of motion, this radial term is of the order of 10^{-3} for the Earth, and is smaller for most other bodies. As the rotation variations of the Earth are of the order of 10^{-5}, this term leads to a tiny change in the geopotential and to an associated displacement of the order of a micrometer.[22]

However, for other rotators the situation may be different. For example, in Phobos, whose libration magnitude is large (about 1 degree), the radial term may cause an equipotential-surface variation of about 10 cm. This magnitude is large enough to be observed by future missions and should be studied in more detail.[23] The emergence of the purely radial deformation gives birth to the zero-degree Love number (Dahlen 1976; Matsuyama and Bills 2010). Using Dahlen's results, Yoder (1982, eqns 21–22) demonstrated that the contribution of the radial part of the centripetal potential to the change in mean motion of Phobos is about 3%, which is smaller than the uncertainty in our knowledge of Phobos' k_2/Q. It should be mentioned, however, that the calculations by Dahlen (1976) and Matsuyama and Bills (2010) were performed for steady (or slowly changing) rotation, and not for libration. So Yoder's application of Dahlen's result to Phobos requires extra justification.

What is important for us here is that the radial term does not interfere with the calculation of the Love numbers of non-zero degree. Being independent of the longitude, this term generates no tidal torque either, provided the obliquity is neglected.

[22] Tim Van Hoolst, private communication.

[23] Tim Van Hoolst, private communication.

B.4 The toroidal term

The inertial term $-\rho\,\dot{\boldsymbol{\omega}} \times \mathbf{r}$ in the equation of motion (159) can be cast into the form

$$- \rho\,\dot{\boldsymbol{\omega}} \times \mathbf{r} = \rho\,\mathbf{r} \times \nabla(\dot{\boldsymbol{\omega}} \cdot \mathbf{r}), \tag{162}$$

whence we see that this term is of a toroidal type. Being almost nil for a despinning primary, this force becomes important for a librating object.

In spherically-symmetric bodies, the toroidal force (162) generates toroidal deformation only. The latter produces neither radial uplifts nor variations of the gravitational potential. Hence its presence does not influence the expressions for the Love numbers associated with vertical displacement (h_l) or the potential (k_l). As this deformation yields no change in the gravitational potential of the tidally-perturbed body, it generates no tidal torque either. Being divergence-free, this deformation entails no contraction or expansion, i.e., it is purely shear. Still, this deformation contributes to dissipation. Besides, since the toroidal forcing yields toroidal deformation, it can, in principle, be associated with a "toroidal Love number".

To estimate the dissipation caused by the toroidal force, Yoder (1982) introduced an equivalent effective torque. He pointed out that the force gets important when the magnitude of the physical libration is comparable to that of the optical libration. According to Yoder (1982), the toroidal force contributes to the change of Phobos' mean motion about 1.6%, which is less than the input from the purely radial part.

C The Andrade and Maxwell models at different frequencies

C.1 Response of a sample obeying the Andrade model

Within the Andrade model, the tangent of the phase lag demonstrates the so-called "elbow dependence". At high frequencies, the tangent of the lag obeys a power law with an exponent equal to $-\alpha$, where $0 < \alpha < 1$. At low frequencies, the tangent of the lag once again obeys a power law, this time though with an exponent $-(1-\alpha)$. This model fits well the behaviour of ices, metals, silicate rocks, and many other materials.

However the applicability of the Andrade law may depend on the intensity of the load and, accordingly, on the damping mechanisms involved. Situations are known, when, at low frequencies, anelasticity becomes much less efficient than viscosity, and the Andrade model approaches the Maxwell model.

C.1.1. The high-frequency band

At high frequencies, expression (85b) gets simplified. In the numerator, the term with $z^{-\alpha}$ dominates: $z^{-\alpha} \gg z^{-1}\,\zeta$, which is equivalent to $z \gg \zeta^{\frac{1}{1-\alpha}}$. In the denominator, the constant term dominates: $1 \gg z^{-\alpha}$, or simply: $z \gg 1$. To know which of the two conditions, $z \gg \zeta^{\frac{1}{1-\alpha}}$ or $z \gg 1$, is stronger, we recall that at high frequencies anelasticity beats viscosity. So the α-term in (81) is large enough. In other words, the Andrade timescale τ_A should be smaller (or, at least, not much higher) than the viscoelastic time τ_M. Accordingly, at high frequencies, ζ is smaller (or, at least, not much higher) than unity. Hence, within the high-frequency band, either the condition $z \gg 1$ is stronger than $z \gg \zeta^{\frac{1}{1-\alpha}}$ or the two conditions are about equivalent. This, along with (86) and (87) enables us to write:

$$\tan\delta \approx (\chi\,\tau_{_A})^{-\alpha}\,\sin\left(\frac{\alpha\,\pi}{2}\right)\,\Gamma(\alpha+1) \quad \text{for} \quad \chi \gg \tau_{_A}^{-1} = \tau_{_M}^{-1}\zeta^{-1}. \tag{163}$$

The tangent being small, the expression for $\sin\delta$ looks identical:

$$\sin\delta \approx (\chi\,\tau_{_A})^{-\alpha}\,\sin\left(\frac{\alpha\,\pi}{2}\right)\,\Gamma(\alpha+1) \quad \text{for} \quad \chi \gg \tau_{_A}^{-1} = \tau_{_M}^{-1}\zeta^{-1}. \tag{164}$$

C.1.2. The intermediate region

In the intermediate region, the behaviour of the phase lag δ depends upon the frequency-dependence of ζ. For example, if there happens to exist an interval of frequencies over which the conditions $1 \gg z \gg \zeta^{\frac{1}{1-\alpha}}$ are obeyed, then over this interval we shall have: $1 \ll z^{-\alpha}$ and $z^{-\alpha} \gg \zeta z^{-1}$. Applying these inequalities to (85b), we see that over such an interval of frequencies $\tan\delta$ will behave as $z^{-2\alpha}\,\tan\left(\frac{\alpha\,\pi}{2}\right)$.

C.1.3 The low-frequency band

At low frequencies, the term $z^{-1}\,\zeta$ becomes leading in the numerator of (85b): $z^{-\alpha} \ll z^{-1}\,\zeta$, which requires $z \ll \zeta^{\frac{1}{1-\alpha}}$. In the denominator, the term with $z^{-\alpha}$ becomes the largest: $1 \ll z^{-\alpha}$, whence $z \ll 1$. Since at low frequencies the viscous term in (81) is larger than the anelastic term, we expect that for these frequencies ζ is larger (at least, not much smaller) than unity. Thence the condition $z \ll 1$ becomes sufficient. Its fulfilment ensures the fulfilment of $z \ll \zeta^{\frac{1}{1-\alpha}}$. Thus we state:

$$\tan\delta \approx (\chi\,\tau_{_A})^{-(1-\alpha)}\,\frac{\zeta}{\cos\left(\frac{\alpha\,\pi}{2}\right)\,\Gamma(\alpha+1)} \quad \text{for} \quad \chi \ll \tau_{_A}^{-1} = \tau_{_M}^{-1}\,\zeta^{-1}. \tag{165}$$

The appropriate expression for $\sin\delta$ will be:

$$\sin\delta \approx 1 - O\left((\chi\,\tau_{_A})^{2(1-\alpha)}\,\zeta^{-2}\right) \quad \text{for} \quad \chi \ll \tau_{_A}^{-1} = \tau_{_M}^{-1}\,\zeta^{-1}, \tag{166}$$

It would be important to emphasise that the threshold $\tau_{_A}^{-1} = \tau_{_M}^{-1}\,\zeta^{-1}$ standing in (163) and (164) is *different* from the threshold $\tau_{_A}^{-1} = \tau_{_M}^{-1}\,\zeta^{-1}$ showing up in (165) and (166), even though these two thresholds are given by the same expression. The reason for this is that the timescales $\tau_{_A}$ and $\tau_{_M}$ are not fixed constants. While the Maxwell time is likely to be a very slow function of the frequency, the Andrade time may undergo a faster change over the transitional region: $\tau_{_A}$ must be larger than $\tau_{_M}$ at low frequencies (so anelasticity yields to viscosity), and must become shorter than or of the order of $\tau_{_M}$ at high frequencies (so anelasticity becomes stronger). This way, the threshold $\tau_{_A}^{-1}$ standing in (165–166) is lower than the threshold $\tau_{_A}^{-1}$ standing in (163–164). The gap between these thresholds is the region intermediate between the two pronounced power laws (163) and (165).

C.1.4 The low-frequency band: a special case, the Maxwell model

Suppose that, below some threshold χ_0, anelasticity quickly becomes *much less* efficient than viscosity. This would imply a steep increase of ζ (equivalently, of $\tau_{_A}$) at low frequencies.

Then, in (85b), we shall have: $1 \gg z^{-\alpha}$ and $z^{-\alpha} \ll \zeta z^{-1}$. This means that, for frequencies below χ_0, the tangent will behave as

$$\tan \delta \approx z^{-1} \zeta = (\chi\, \tau_{_M})^{-1} \quad \text{for} \quad \chi \ll \chi_0. \tag{167}$$

the well-known viscous scaling law for the lag.

The study of ices and minerals under weak loads (Castillo-Rogez et al. 2011; Castillo-Rogez and Choukroun 2010) has not shown such an abrupt vanishing of anelasticity. However, Karato and Spetzler (1990) point out that this should be happening in the Earth's mantle, where the loads are much higher and anelasticity is caused by unpinning of dislocations.

C.2 The behaviour of $|k_l(\chi)| \ \sin \epsilon_l(\chi) = -\mathcal{I}m\left[\bar{k}_l(\chi)\right]$ within the Andrade and Maxwell models

As we explained in Sect. 4.1, products $k_l(\chi_{lmpq}) \sin \epsilon_l(\chi_{lmpq})$ enter the $lmpq$ term of the Darwin–Kaula series for the tidal potential, force, and torque. Hence the importance to know the behaviour of these products as functions of the tidal frequency χ_{lmpq}.

C.2.1 Prefatory algebra

It ensues from (64) that

$$\bar{k}_l(\chi) = \frac{3}{2\,(l-1)}\ \frac{\left(\mathcal{R}e\left[\bar{J}(\chi)\right]\right)^2 + \left(\mathcal{I}m\left[\bar{J}(\chi)\right]\right)^2 + A_l\ J\ \mathcal{R}e\left[\bar{J}(\chi)\right] + i\ A_l\ J\ \mathcal{I}m\left[\bar{J}(\chi)\right]}{\left(\mathcal{R}e\left[\bar{J}(\chi)\right] + A_l\ J\right)^2 + \left(\mathcal{I}m\left[\bar{J}(\chi)\right]\right)^2}, \tag{168}$$

whence

$$|\bar{k}_l(\chi)| \ \sin \epsilon_l(\chi) = -\mathcal{I}m\left[\bar{k}_l(\chi)\right] = \frac{3}{2\,(l-1)}\ \frac{-A_l\, J \mathcal{I}m\left[\bar{J}(\chi)\right]}{\left(\mathcal{R}e\left[\bar{J}(\chi)\right] + A_l\ J\right)^2 + \left(\mathcal{I}m\left[\bar{J}(\chi)\right]\right)^2}, \tag{169}$$

$J = J(0) \equiv 1/\mu = 1/\mu(0)$ being the unrelaxed compliance (the inverse of the unrelaxed shear modulus μ). For an Andrade material, the compliance $\bar{J}$ in the frequency domain is rendered by (82). Its imaginary and real parts are given by (83–84). It is then easier to rewrite (169) as

$$|\bar{k}_l(\chi)| \ \sin \epsilon_l(\chi) = \frac{3\, A_l}{2\,(l-1)}$$
$$\times \frac{\zeta\, z^{-1} + z^{-\alpha}\ \sin\left(\frac{\alpha\,\pi}{2}\right)\ \Gamma(\alpha+1)}{\left[A_l + 1 + z^{-\alpha}\ \cos\left(\frac{\alpha\,\pi}{2}\right)\ \Gamma(\alpha+1)\right]^2 + \left[\zeta\, z^{-1} + z^{-\alpha}\ \sin\left(\frac{\alpha\,\pi}{2}\right)\ \Gamma(1+\alpha)\right]^2}, \tag{170}$$

where

$$z \equiv \chi\, \tau_{_A} = \chi\, \tau_{_M}\, \zeta \tag{171}$$

and

$$\zeta \equiv \tau_{_A}/\tau_{_M}. \tag{172}$$

For $\beta \to 0$, i.e., for $\tau_{_A} \to \infty$, (170) coincides with the appropriate expression for a spherical Maxwell body.

C.2.2 The high-frequency band

Within the upper band, the term with $z^{-\alpha}$ dominates the numerator, while $A_l + 1$ dominates the denominator. The dominance of $z^{-\alpha}$ in the numerator requires that $z \gg \zeta^{\frac{1}{1-\alpha}}$, which is:

$$\chi\;\tau_{_M} \gg \zeta^{\frac{\alpha}{1-\alpha}}\;.$$

The dominance of $A_l + 1$ in the denominator means: $z \gg \;(A_l + 1)^{-1/\alpha}$, which is the same as

$$\chi\;\tau_{_M} \gg \zeta^{-1}\,(A_l + 1)^{-1/\alpha}\;.$$

The dominance of $A_l + 1$ also demands that $\zeta\;z^{-1} \ll (A_l + 1)$, which is:

$$\chi\;\tau_{_M} \gg (A_l + 1)^{-1}.$$

Simultaneous fulfilment of all these conditions entails:

$$|\bar{k}_l(\chi)|\;\sin\epsilon_l(\chi) \approx \frac{3}{2\,(l-1)}\;\frac{A_l}{(A_l+1)^2}\;\sin\left(\frac{\alpha\,\pi}{2}\right)\Gamma(\alpha+1)\,\zeta^{-\alpha}\left(\tau_{_M}\;\chi\right)^{-\alpha},\quad \text{for}\quad \chi \gg \chi_{_{HI}}, \tag{173a}$$

where the boundary between the high and intermediate frequencies, $\chi_{_{HI}}$, is defined as the quantity largest among $\tau_{_M}^{-1}\,\zeta^{\frac{\alpha}{1-\alpha}}$, $\tau_{_M}^{-1}\,\zeta^{-1}\,(A_l + 1)^{-1/\alpha}$, and $\tau_{_M}^{-1}\,(A_l + 1)^{-1}$.

For small terrestrial objects, $A_l \gg 1$ (Efroimsky 2012). As at high frequencies anelasticity beats viscosity, ζ may be of order unity or slightly less than unity (not by orders of magnitude, likely). Hence *for small bodies and small terrestrial planets* $\chi_{_{HI}} = \tau_{_M}^{-1}\,\zeta^{\frac{\alpha}{1-\alpha}}$.

For solid bodies much larger than the Earth, $A_l \ll 1$ (see Efroimsky 2012). Hence *for large terrestrial exoplanets* $\chi_{_{HI}} = \tau_{_M}^{-1}\,\zeta^{-1}$.

C.2.3 The intermediate band

Within the intermediate band, the term $\zeta\;z^{-1}$ takes over in the numerator, while $A_l + 1$ still dominates in the denominator. The dominance of $\zeta\;z^{-1}$ in the numerator implies that $z \ll \zeta^{\frac{1}{1-\alpha}}$, which is equivalent to

$$\chi\;\tau_{_M}\;\ll\;\zeta^{\frac{\alpha}{1-\alpha}}\;.$$

The dominance of $A_l + 1$ in the denominator requires, as we just saw:

$$\chi\;\tau_{_M} \gg \zeta^{-1}\,(A_l + 1)^{-1/\alpha}$$

and

$$\chi\;\tau_{_M} \gg (A_l + 1)^{-1}.$$

We are considering the band where the efficiency of viscosity is either comparable or exceeds that of anelasticity. Accordingly, ζ is about or larger than unity. Then, of the two above-written inequalities, the latter is stronger, for any A_l. Thus we obtain:

$$|\bar{k}_l(\chi)|\ \sin\epsilon_l(\chi) \approx \frac{3}{2\,(l-1)}\ \frac{A_l}{(A_l+1)^2}\ \left(\tau_{_M}\,\chi\right)^{-1}, \quad \text{for} \quad \tau_{_M}^{-1}\,\zeta^{\,\frac{\alpha}{1-\alpha}} \gg \chi \gg \tau_{_M}^{-1}(A_l+1)^{-1}. \tag{173b}$$

C.2.4 The low-frequency band

For frequencies lower than $\tau_{_M}^{-1}(A_l+1)^{-1}$, a self-gravitating Andrade body renders the same frequency-dependency as a self-gravitating Maxwell body:

$$|\bar{k}_l(\chi)|\ \sin\epsilon_l(\chi) \approx \frac{3}{2\,(l-1)}\,A_l\,\tau_{_M}\,\chi, \quad \text{for} \quad \tau_{_M}^{-1}\,A_l^{-1} \gg \chi. \tag{173c}$$

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